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L_∞ Markov and Bernstein Inequalities for Erdős Weights

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Recently, weighted Markov and Bernstein inequalities have been established for large classes of Freud weights, that is, weights of the form $W(x) := e^{-Q(x)}$, where $Q(x)$ is even and of smooth polynomial growth at infinity. In this paper, we consider Erdős weights, which have the form $W(x) := e^{-Q(x)}$, where $Q(x)$ is even and of faster than polynomial growth at infinity. For a large class of Erdős weights, we establish the Markov type inequality

$$\|P'W\|_{\mathbb{R}} \leqslant CQ'(a_n)\,\|PW\|_{\mathbb{R}}, \tag{1}$$

for $n \geqslant 1$ and P any polynomial of degree at most n. Here the norm is the sup norm, and C is independent of n and P, while a_n is the Mhaskar–Rahmanov–Saff number, that is, it is the positive root of the equation

$$n = \frac{2}{\pi}\int_0^1 a_n t Q'(a_n t)\, dt/\sqrt{1-t^2}. \tag{2}$$

For example, we consider $Q(x) := \exp_k(|x|^\alpha)$, where $\alpha > 0$, and where $\exp_k$ denotes the kth iterated exponential, and give a more explicit formulation of (1). We also establish Bernstein type inequalities that for part of the range $(-\infty, \infty)$ improve on (1).

1. INTRODUCTION AND STATEMENT OF RESULTS

In converse or Bernstein type theorems on the degree of approximation by polynomials, a crucial role is played by Markov–Bernstein inequalities, which estimate the derivative of a polynomial in terms of its norm. In recent years, much effort has been devoted to establishing such inequalities in weighted norms over $\mathbb{R}$. See [20] for an entertaining introduction, [4]

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for the relevant approximation theorems, and [12, 21] for the most recent and up to date L_∞ results. For the most up to date treatments of L_p and Orlicz space norms, see especially [15, 21] and also [7, 11, 20].

To elaborate the discussion, we need some notation. Throughout, $\mathscr{P}_n$ denotes the class of real polynomials of degree at most n, and $\|\cdot\|_{\mathscr{S}}$ denotes the L_∞ norm over any measurable $\mathscr{S} \subset \mathbb{R}$. Further, $C, C_1, C_2, \ldots,$ denote positive constants independent of n, $P \in \mathscr{P}_n$, and $x \in \mathbb{R}$. The same symbol does not necessarily denote the same constant in different occurrences. Finally, we use the usual o, O notation, and $\sim$ in the following sense: If $\{c_n\}_1^\infty$ and $\{d_n\}_1^\infty$ are sequences of real numbers, we write

$$c_n \sim d_n,$$

if there exist C_1 and C_2 such that for the relevant range of n,

$$C_1 \leqslant c_n/d_n \leqslant C_2.$$

Similar notations will be used for functions and sequences of functions.

The classical inequality of Markov [3, p. 91] is

$$\|P'\|_{[-1,1]} \leqslant n^2 \|P\|_{[-1,1]}, \qquad P \in \mathscr{P}_n. \tag{1.1}$$

Essentially the most general analogue of (1.1) for Freud weights, that is, weights of the form $W := e^{-Q}$, where $Q(x)$ is even and of smooth polynomial growth at infinity, is the following [12, Theorem 1.1]:

THEOREM 1.1. *Let $W(x) := e^{-Q(x)}$, where $Q(x)$ is even, continuous in $\mathbb{R}$, $Q(0) = 0$, $Q''(x)$ is continuous in $(0, \infty)$, $Q'(x)$ is positive in $(0, \infty)$, and for some $C_1, C_2 > 0$,*

$$C_1 \leqslant (xQ'(x))'/Q'(x) \leqslant C_2, \qquad x \in (0, \infty). \tag{1.2}$$

Then there exists $C_3 > 0$ such that for $n = 1, 2, 3, \ldots,$ and $P \in \mathscr{P}_n$,

$$\|P'W\|_{\mathbb{R}} \leqslant \left\{ \int_1^{C_3 n} ds/Q^{[-1]}(s) \right\} \|PW\|_{\mathbb{R}}, \tag{1.3}$$

where $Q^{[-1]}$ is the inverse function of $Q(x)$, satisfying

$$Q^{[-1]}(Q(s)) = s, \qquad s \in (0, \infty). \tag{1.4}$$

In the important special case

$$W_\alpha(x) := \exp(-|x|^\alpha), \qquad x \in \mathbb{R},\ \alpha > 0,$$

Theorem 1.1 yields for $n \geqslant 1$ and for $P \in \mathscr{P}_n$ and some C,

$$\|P'W_\alpha\|_{\mathbb{R}} \leqslant C\|PW_\alpha\|_{\mathbb{R}} \begin{cases} n^{1-1/\alpha}, & \alpha > 1, \\ \log(n+1), & \alpha = 1, \\ 1, & 0 < \alpha < 1. \end{cases} \tag{1.5}$$

For $\alpha \geqslant 2$, Freud [8] established (1.5), while Levin and Lubinsky [10, 11] treated the cases $1 < \alpha < 2$, as well as related weights. For $0 < \alpha \leqslant 1$, (1.5) was established by Nevai and Totik [21], and they considered more general weights similar to W_α, $0 < \alpha < 1$. For fixed finite intervals $[a, b]$ and $n \geqslant N(a, b)$, Dzrbasyan [5] established similar inequalities for more general weights, though his constants depend on a, b.

The condition (1.2) was heavily used in [12] and forces $Q(x)$ to be of polynomial growth at infinity. In this paper, we consider the case where $Q(x)$ is of faster than polynomial growth at infinity. We call $W := e^{-Q}$, with such a Q, an *Erdős weight*, for Erdős was the first to consider them [6], obtaining the contracted zero distribution of their orthogonal polynomials. Asymptotics for the recurrence coefficients associated with their orthogonal polynomials were obtained in [9]. A typical example is

$$W_{k,\alpha}(x) := \exp(-\exp_k(|x|^\alpha)), \qquad x \in \mathbb{R}, \tag{1.6}$$

where $\alpha > 0$, k is a positive integer, and $\exp_k$ is the kth iterated exponential:

$$\exp_1(x) := \exp(x), \qquad x \in \mathbb{R},$$

$$\exp_k(x) := \exp(\exp_{k-1}(x)), \qquad x \in \mathbb{R},\ k = 2, 3, 4, \ldots .$$

The Markov inequalities for Erdős weights are somewhat more enigmatic than those for Freud weights, and are closer to those for weights on $[-1, 1]$. The quantity

$$\int_1^{C_3 n} ds/Q^{[-1]}(s)$$

in the right-hand side of (1.3) is $o(n)$ as $n \to \infty$, while n^2 in (1.1) grows much faster than n. For Erdős weights, the dependence on n of the right-hand sides of the Markov inequalities may also grow faster than n. Perhaps this should not be surprising, for Erdős weights decay much more rapidly than Freud weights, and in this and other respects are like weights on $[-1, 1]$ [6]. To describe the inequalities, we need:

DEFINITION 1.2. Let $W(x) := e^{-Q(x)}$, where $Q(x)$ is even and continuous in $\mathbb{R}$, $Q'(x)$ exists in $(0, \infty)$, and $xQ'(x)$ is increasing in $(0, \infty)$ with limits 0 and ∞ at 0 and ∞, respectively. For $u > 0$, we define the

Mhaskar–Rahmanov–Saff number $a_u = a_u(W)$ to be the positive root of the equation

$$u = \frac{2}{\pi} \int_0^1 a_u t Q'(a_u t)(1-t^2)^{-1/2}\, dt. \tag{1.7}$$

It is easily seen under the conditions in Definition 1.2 that for all $u > 0$, a_u exists and is unique.

The number a_n (for positive integer n) appears first in [17–19, 22]. Its importance lies in the following identity: If $W := e^{-Q}$, and Q is even in $\mathbb{R}$, then under mild conditions on Q' [16, 19], we have for all $P \in \mathscr{P}_n$,

$$\|PW\|_{\mathbb{R}} = \|PW\|_{[-a_n, a_n]}, \tag{1.8}$$

and $[-a_n, a_n]$ is essentially the smallest finite interval for this result to hold [16, 19]. Typically, a_n exhibits the following rate of growth:

$$a_n \sim Q^{[-1]}(n), \qquad n \to \infty.$$

One of our main results is the following Markov type inequality:

THEOREM 1.3 (Markov Inequality). *Let* $W(x) := e^{-Q(x)}$, *where* $Q(x)$ *is even and continuous in* $\mathbb{R}$, $Q''(x)$ *is continuous in* $(0, \infty)$,

$$Q'(x) > 0, \qquad x \in (0, \infty), \tag{1.9}$$

and

$$\chi(x) := (xQ'(x))'/Q'(x), \qquad x \in (0, \infty), \tag{1.10}$$

is positive and increasing in $(0, \infty)$ *with* $\chi(0+) > 0$ *and*

$$\lim_{x \to \infty} \chi(x) = \infty, \tag{1.11}$$

while

$$\chi(x) = O(Q'(x)^{1/2}), \qquad x \to \infty. \tag{1.12}$$

Then there exists C such that for $n \geq 1$, *and* $P \in \mathscr{P}_n$,

$$\|P'W\|_{\mathbb{R}} \leq CQ'(a_n)\, \|PW\|_{\mathbb{R}}. \tag{1.13}$$

Remarks. (i) While (1.11) ensures that $Q(x)$ grows faster as $x \to \infty$ than any polynomial (in comparison to (1.2), which ensures polynomial growth), (1.12) is a very weak regularity condition. In fact, for any $Q(x)$ satisfying the conditions of Theorem 1.3 (except possibly (1.12)), and for any $\varepsilon > 0$,

$$\chi(x) < \varepsilon(Q'(x))^{\varepsilon} \qquad \text{on average.}$$

More precisely, if meas denotes linear Lebesgue measure, it is not difficult to show that

$$\text{meas}\{x \geqslant r: \chi(x) \geqslant \varepsilon(Q'(x))^{\varepsilon}\} \to 0 \qquad \text{as} \quad r \to \infty.$$

In fact, one typically has much more: For each $\varepsilon > 0$,

$$\chi(x) = O([\log Q'(x)]^{1+\varepsilon}) \qquad \text{as} \quad x \to \infty.$$

(ii) If, for example, $\alpha > 0$, k is a positive integer, and (see (1.6))

$$Q(x) := \exp_k(|x|^{\alpha}), \qquad x \in \mathbb{R}, \tag{1.14}$$

while $W_{k,\alpha} := e^{-Q}$, then all the conditions of Theorem 1.3 are satisfied, and

$$\chi(x) = \{\alpha \log Q(x) \log_2 Q(x) \cdots \log_k Q(x)\}(1 + o(1)) \qquad \text{as} \quad x \to \infty,$$

where $\log_k$ denotes the kth iterated logarithm, that is,

$$\log_1 x := \log x, \qquad x > 0,$$
$$\log_k x := \log_{k-1}(\log x), \qquad x > \exp_{k-1}(0),\ k = 2, 3, 4, \ldots .$$

Further, a straightforward, but lengthy computation involving Laplace's method shows that

$$a_n^{\alpha} = \log_{k-1}\left(\log n - \frac{1}{2}\sum_{j=2}^{k+1} \log_j n + O(1)\right), \qquad n \to \infty, \tag{1.15}$$

and

$$\begin{aligned} Q'(a_n) &\sim n\chi(Q^{[-1]}(n))^{1/2}/Q^{[-1]}(n) \\ &\sim n\left[\prod_{j=1}^{k} \log_j n\right]^{1/2} (\log_k n)^{-1/\alpha}, \qquad n \to \infty. \end{aligned} \tag{1.16}$$

Note that for $\alpha > 2$ and $k \geqslant 1$,

$$\lim_{n \to \infty} Q'(a_n)/n = \infty.$$

It follows from (1.16) that Theorem 1.3 improves on some results in the literature. In [13, Theorem 3.5, (3.20)], it was shown that for $n \geqslant n_0$ and $P \in \mathscr{P}_n$,

$$\|P'W_{k,\alpha}\|_{\mathbb{R}} \leqslant Cn\left[\prod_{j=1}^{k} \log_j n\right]^2 (\log_k n)^{-1/\alpha} \|PW_{k,\alpha}\|_{\mathbb{R}},$$

and conjectured that the 2 may be replaced by $\frac{1}{2}$. This conjecture is confirmed by (1.16). In [1], a former student of the author considered $W_{1,2}$ and obtained a slight improvement of (3.20) in [13], replacing the 2 above by 1.

(iii) Concerning the rate of growth of $Q'(a_n)$ in the general case treated by Theorem 1.3, we note that (see Lemma 2.2(a), (c) below)

$$\lim_{n\to\infty} Q'(a_n)/(n/a_n) = \infty, \tag{1.17}$$

but

$$Q'(a_n)/(n/a_n) = O(\chi(a_n)^{1/2}), \qquad n\to\infty. \tag{1.18}$$

Under additional conditions on Q, one can replace the O in (1.18) by $\sim$, and one can show that

$$Q'(a_n) \sim n\chi(Q^{[-1]}(n))^{1/2}/Q^{[-1]}(n), \qquad n\to\infty.$$

(iv) It seems certain that Theorem 1.3 is sharp in the sense that $Q'(a_n)$ provides the correct rate of growth in n. Although we do not prove this formally, we shall provide the following motivation: Let $T_n^*(x)$ denote that monic polynomial of degree n for which

$$\|T_n^* W\|_{\mathbb{R}} = \min\{\|PW\|_{\mathbb{R}} : P \text{ monic, } P\in\mathscr{P}_n\}.$$

It is known that $|T_n^* W|$ attains its maximum at at least $n+1$ points, of which ξ_n, say, is the largest [16, 19]. Then

$$\begin{aligned}\|T_n^{*\prime} W\|_{\mathbb{R}} &\geq |T_n^{*\prime} W|\,(\xi_n)\\ &= |Q'(\xi_n)(T_n^* W)(\xi_n) + (T_n^* W)'\,(\xi_n)|\\ &= Q'(\xi_n)\,\|T_n^* W\|_{\mathbb{R}}.\end{aligned}$$

We believe that under the conditions of Theorem 1.3,

$$\lim_{n\to\infty} Q'(\xi_n)/Q'(a_n) = 1, \tag{1.19}$$

and hope to prove this in a forthcoming paper. Certainly (1.19) is true in the case of Freud weights [16], but is a little deeper for Erdős weights.

(v) Despite the different appearances of Theorems 1.1 and 1.3, their results do agree in form: For Freud weights for which $Q(x)$ grows at least as fast as $|x|^\alpha$, some $\alpha>1$, one can show that

$$\int_1^{C_3 n} ds/Q^{[-1]}(s) \sim Q'(a_n) \qquad \text{as} \quad n\to\infty.$$

(vi) Theorem 1.3 remains valid if all the conditions on Q (other than continuity) hold only for large x. One needs then to modify, in an obvious way, the definition of a_n.

(vii) For more general W than considered here, Corollary 3.2 in [13, p. 348] shows that for each fixed $0<\delta<1$, there exists $C=C(\delta, W)$ such that

$$\|P'W\|_{[-\delta a_n, \delta a_n]} \leqslant C(n/a_n) \|PW\|_{\mathbb{R}}, \tag{1.20}$$

$P \in \mathscr{P}_n$, $n \geqslant 1$. In view of (1.17), this improves on (1.13) for the interval $[-\delta a_n, \delta a_n]$. Such an improvement is explained by our Bernstein inequality below.

Recall the classical Bernstein inequality [3, pp. 89–91], which states that

$$|P'(x)| \leqslant n(1-x^2)^{-1/2} \|P\|_{[-1,1]}, \qquad x \in (-1,1),\ P \in \mathscr{P}_n. \tag{1.21}$$

For $|x| \leqslant \delta < 1$, this yields, for n large enough, better results than Markov's (1.1). For Erdős weights, (1.20) provides the corresponding improvement of (1.13), for $|x| \leqslant \delta a_n$, any $0<\delta<1$. As x increases towards a_n, the dependence on n seems first to grow faster than n/a_n, but for x very close to a_n, grows slower than n/a_n. The precise description is quite complicated.

First, however, we recall from [12, Theorem 1.3], for comparison, part of the Bernstein inequality there:

THEOREM 1.4. *Let $W(x)$ be as in Theorem 1.1, and let $a_n = a_n(W)$ for $n = 1, 2, 3, \ldots$. Let $0<\eta<1$. Then for $n \geqslant C_3$, $P \in \mathscr{P}_n$, and $|x| > \eta a_n$,*

$$|(PW)'(x)| \leqslant C_4 \|PW\|_{\mathbb{R}} (n/a_n) \max\{n^{-2/3}, 1-|x|/a_n\}^{1/2}. \tag{1.22}$$

As remarked in [12], it is essential that we consider $(PW)'$ rather than $P'W$ for the Bernstein inequality. We believe that Theorems 1.4 and 1.5 may play a role in establishing bounds for orthogonal polynomials generalizing those in [2]. Following is our

THEOREM 1.5 (Bernstein Inequality). *Let $W(x)$ be as in Theorem 1.3, with the additional restrictions that $Q'(x)$ is continuous in $\mathbb{R}$, and that (1.12) holds with $\frac{1}{2}$ replaced by $\frac{1}{12}$. Let $\xi > 0$, and for $n \geqslant 1$, let*

$$\psi_n(x) := \int_{\xi/a_n}^{1} (1-s)^{-1/2} \frac{a_n x Q'(a_n x) - a_n s Q'(a_n s)}{a_n x - a_n s}\, ds, \qquad x \in [0,1], \tag{1.23}$$

and let

$$A_n^* := n^{-1} \int_{1/2}^{1} (1-s)^{-1/2} (a_n s)^2 Q''(a_n s)\, ds. \tag{1.24}$$

Then for $n \geqslant C_1$, $P \in \mathscr{P}_n$, and any $r > 0$,

$$|(PW)'(x)| \leqslant C\|PW\|_{\mathbb{R}} \times \begin{cases} (1-|x/a_n|)^{-1} \int_{|x/a_n|}^{1} \psi_n(t)(1-t)^{1/2}\,dt, \\ \qquad |x/a_n| \leqslant 1 - r(nA_n^*)^{-2/3}, \\ (nA_n^*)^{2/3}/a_n, \\ \qquad |x/a_n| \geqslant 1 - r(nA_n^*)^{-2/3}. \end{cases} \tag{1.25}$$

In particular, this implies that given any $0 < \delta < 1$,

$$|(PW)'(x)|_{\mathbb{R}} \leqslant C\|PW\|_{\mathbb{R}}(n/a_n), \qquad |x| \leqslant a_n(1-\delta),\ P \in \mathscr{P}_n. \tag{1.26}$$

Remarks. (i) We do not know of any simpler way to express (1.25) for general Erdős weights. For Freud weights, an essential simplification is that

$$A_n^* \sim 1; \qquad \psi_n(x) \sim n/a_n \qquad \text{uniformly for } |x| \leqslant 1,$$

and one can easily show that the right-hand side of (1.25) reduces to the right-hand side of (1.22). By contrast for Erdős weights,

$$\lim_{n \to \infty} A_n^* = \infty,$$

and

$$\psi_n(x)/(n/a_n)$$

is unbounded. Nevertheless A_n^* grows slowly, and (Lemma 3.2(f) below)

$$A_n^* = O(\chi(a_n)),$$

while for Q of (1.14),

$$A_n^* \sim \chi(a_n) \sim \chi(Q^{[-1]}(n)) \sim \prod_{j=1}^{k} \log_j n, \qquad n \to \infty.$$

(ii) The condition that Q' be continuous in $\mathbb{R}$ is imposed purely for W' to exist in $\mathbb{R}$. If, for example, $Q'(0)$ does not exist, but the other conditions are satisfied, then (1.25) remains valid for $x \neq 0$.

(iii) We believe the above result is sharp with respect to the dependence on n: The estimates arise from solutions of certain integral equations that are now known to play a fundamental role in the majorization of weighted polynomials, and asymptotics of orthogonal polynomials [16, 17, 23].

(iv) Theorem 1.5 is consistent with Theorem 1.3, in the sense that the right-hand side of (1.25) is bounded above by $CQ'(a_n) \|PW\|_{\mathbb{R}}$.

(v) For $|x| > a_n$, (1.25) admits a substantial improvement—see the proof of Theorem 1.5—but we omitted this from the statement above since that range of x is not so important in applications.

This paper is organized as follows: In Section 2, we present three preliminary technical lemmas. In Section 3, we estimate $U_n(t)$, a function that arises in the majorization of extremal polynomials. In Section 4, we prove Theorems 1.3 and 1.5. On a first reading, the reader should perhaps start with the basic Lemma 4.1, which uses Cauchy's integral formula for derivatives to estimate $(PW)'$. After reading Section 4, and then Section 3, the reader can turn to Section 2.

2. Preliminary Lemmas

We shall say a function f: $[0, \infty) \to [0, \infty)$ is *quasi-increasing* if there exists $C > 0$ such that

$$f(x) \leqslant Cf(y), \qquad 0 \leqslant x \leqslant y < \infty.$$

This is trivially true if f is increasing. In our proofs, we shall initially use slightly different assumptions from those in Theorem 1.3, and shall ultimately replace the given weight by a slightly different one. This is necessitated by the occasionally difficult behaviour of Q' at 0.

Lemma 2.1. *Let $W(x) := e^{-Q(x)}$, where Q is even and continuous in $\mathbb{R}$, Q'' is continuous in $(0, \infty)$,*

$$Q'(x) > 0, \qquad x \in (0, \infty), \tag{2.1}$$

while

$$(xQ'(x))' > 0, \qquad x \in (0, \infty). \tag{2.2}$$

Further assume that

$$\chi(x) := (xQ'(x))'/Q'(x), \qquad x \in (0, \infty), \tag{2.3}$$

is bounded below by a positive number in $(0, \infty)$, is quasi-increasing in $(0, \infty)$, and increasing for large x, with

$$\lim_{x \to \infty} \chi(x) = \infty. \tag{2.4}$$

Then:

(a) *Given $r>0$, there exists C such that*

$$Q^{(j)}(x) \geqslant x^r, \qquad x \geqslant C,\ j=0, 1, 2. \tag{2.5}$$

(b) *$Q''(x)$ and $Q'(x)/x$ are increasing for large enough x.*

(c) *There exists C such that for $L \geqslant 1$ and $x \in (0, \infty)$,*

$$L^{\chi(x)/C-1} \leqslant Q'(Lx)/Q'(x) \leqslant L^{C\chi(Lx)-1}. \tag{2.6}$$

(d) *Also*

$$\lim_{x \to 0+} xQ'(x) = 0. \tag{2.7}$$

(e) *For $j=0, 1, 2$, and each fixed $L>1$,*

$$\lim_{x \to \infty} Q^{(j)}(Lx)/Q^{(j)}(x) = \infty. \tag{2.8}$$

(f) *For $j=0, 1$,*

$$\lim_{x \to \infty} xQ^{(j+1)}(x)/Q^{(j)}(x) = \infty. \tag{2.9}$$

(g) *Given $r>1$, there exist C_1 and C_2 such that*

$$\chi(x) \leqslant C_1 + C_2 \log\{Q'(rx)/Q'(x)\}, \qquad x \in (0, \infty). \tag{2.10}$$

(h) *If also Q'' is continuous in $\mathbb{R}$, then there exist C and $s>0$ such that*

$$Q'(x)/x \leqslant CQ'(y)/y, \qquad 0<x \leqslant y,\ y \geqslant s, \tag{2.11}$$

and

$$|Q^{(j)}(x)| \leqslant C|Q^{(j)}(y)|, \qquad 0<x \leqslant y,\ y \geqslant s,\ j=1, 2. \tag{2.12}$$

Proof. (a) Now, from (2.3),

$$\chi(x) = xQ''(x)/Q'(x) + 1; \tag{2.13}$$

so (2.4) yields, for t large enough, say for $t \geqslant C_1$,

$$Q''(t)/Q'(t) \geqslant 2r/t.$$

Integrating from $t=C_1$ to $t=x$ yields

$$\log\{Q'(x)/Q'(C_1)\} \geqslant 2r \log(x/C_1),$$

or

$$Q'(x) \geqslant Q'(C_1)(x/C_1)^{2r}.$$

Then (2.5) follows for $j=1$ and $x \geqslant C$, some large enough C. Integrating (2.5) for $j=1$ yields (2.5) for $j=0$ and x large enough. Finally, since (2.4) and (2.13) show that

$$Q''(x) \geqslant Q'(x)/x, \qquad x \text{ large enough,}$$

(2.5) follows also for $j=2$.

(b) Now,

$$\begin{aligned}(Q'(x)/x)' &= (xQ''(x) - Q'(x))/x^2 \\ &= Q'(x)(\chi(x)-2)/x^2 > 0,\end{aligned}$$

x large enough, so $Q'(x)/x$ is increasing for x large enough. Since from (2.13),

$$Q''(x) = (\chi(x)-1)(Q'(x)/x),$$

and $\chi(x)$ is increasing for large enough x, the same is true for Q''.

(c) Now, for $x>0$ and $L \geqslant 1$,

$$\begin{aligned}\{LxQ'(Lx)\}/\{xQ'(x)\} &= \exp\left(\int_x^{Lx} (uQ'(u))'/(uQ'(u))\,du\right) \\ &= \exp\left(\int_x^{Lx} \chi(u)/u\,du\right) \\ &\begin{cases} \leqslant \exp\left(C\chi(Lx)\int_x^{Lx} du/u\right), \\ \geqslant \exp\left(C^{-1}\chi(x)\int_x^{Lx} du/u\right), \end{cases}\end{aligned}$$

as χ is quasi-increasing. Then (2.6) follows.

(d) Choose fixed $a>0$, and let $x \in (0, a)$. From (2.6),

$$xQ'(x) \leqslant aQ'(a)(x/a)^{\chi(x)/C}.$$

Since $\chi(x)$ is bounded below by a positive number, we may let $x \to 0+$.

(e) For $j=1$, (2.8) follows from (2.6) and (2.4). For $j=2$,

$$Q''(Lx)/Q''(x) = \left\{\frac{\chi(Lx)-1}{L(\chi(x)-1)}\right\}\{Q'(Lx)/Q'(x)\} \to \infty \qquad \text{as} \quad x \to \infty,$$

since L is fixed, and $\chi(\cdot)$ is quasi-increasing. This establishes (2.8) for $j=2$

also. To prove (2.8) for $j=0$, we note first that given $r>0$, there exists C such that

$$Q'(Lt) \geqslant rQ'(t), \qquad t \geqslant C.$$

Then as $Q(x)$ is positive for large enough x, say for $x \geqslant C$, we have

$$\begin{aligned} Q(Lx) &= \int_C^x LQ'(Lt)\,dt + Q(LC) \\ &\geqslant Lr\int_C^x Q'(t)\,dt \\ &= Lr(Q(x)-Q(C)) \geqslant LrQ(x)/2, \end{aligned}$$

x large enough. As r may be chosen arbitrarily large, (2.8) follows for $j=0$.

(f) For $j=1$, (2.9) follows from (2.4) (see (2.13)). For $j=0$, we have for x large enough,

$$\begin{aligned} Q(x) &= Q(x/2) + x\int_{1/2}^1 Q'(ux)\,du \\ &\leqslant Q(x)/2 + x\int_{1/2}^1 Q'(ux)\,du, \end{aligned}$$

by (2.8) with $j=0$, and x large enough. Then

$$Q(x)/(xQ'(x)) \leqslant 2\int_{1/2}^1 (Q'(ux)/Q'(x))\,du,$$

for x large enough. Here, for each fixed $u \in [\frac{1}{2}, 1)$, (2.8) with $j=1$ yields

$$\lim_{x\to\infty} Q'(ux)/Q'(x) = 0.$$

Further, as (2.5) shows $Q'(s)$ is increasing for s large enough, we have

$$Q'(ux)/Q'(x) \leqslant 1, \qquad u \in [\tfrac{1}{2}, 1], \ x \text{ large enough.}$$

Then Lebesgue's Dominated Convergence Theorem yields, as required,

$$\lim_{x\to\infty} Q(x)/(xQ'(x)) = 0.$$

(g) Since $\chi(x)$ is quasi-increasing in $(0, \infty)$, for $x \in (0, \infty)$, we have

$$\int_x^{rx} \chi(u)\,du \geqslant C(r-1)x\chi(x),$$

and

$$\int_x^{rx} \chi(u)\,du \leqslant (r-1)\,x + rx \int_x^{rx} Q''(u)/Q'(u)\,du$$
$$= rx[(1-r^{-1}) + \log\{Q'(rx)/Q'(x)\}].$$

Hence

$$\chi(x) \leqslant \frac{r}{C(r-1)}\,[(1-r^{-1}) + \log\{Q'(rx)/Q'(x)\}].$$

(h) Since $Q'(x)/x$, $Q'(x)$, and $Q''(x)$ are increasing in $[a, \infty)$, some $a > 0$, it suffices to deal with the interval $[0, a]$. First, $Q'(0) = 0$ since Q' is odd and continuous at 0. Then

$$Q'(x) = \int_0^x Q''(u)\,du \leqslant x\|Q''\|_{[0,a]}, \qquad x \in [0, a];$$

so $Q'(x)/x$ is bounded in $(0, a]$. Since $Q'(a)/a > 0$, we obtain

$$Q'(x)/x \leqslant CQ'(a)/a, \qquad x \in (0, a].$$

Then (2.11) follows. To prove (2.12), one uses the continuity of $Q^{(j)}$, $j = 1, 2$, and the fact that $Q^{(j)}(a) > 0$ if a is large enough. ∎

Next, a lemma about a_n:

LEMMA 2.2. *Let $W(x)$ be as in Lemma 2.1.*

(a) *Then*

$$\lim_{n\to\infty} a_n^j Q^{(j)}(a_n)/n = \begin{cases} 0, & j = 0, \\ \infty, & j = 1, 2. \end{cases} \tag{2.14}$$

(b) *Uniformly for x in compact subsets of* $(0, 1)$, *we have*

$$\lim_{n\to\infty} a_n^j Q^{(j)}(a_n x)/n = 0, \qquad j = 0, 1, 2. \tag{2.15}$$

(c) *For $j = 1, 2$ and n large enough,*

$$a_n^j Q^{(j)}(a_n)/n \leqslant C\chi(a_n)^{j-1/2}. \tag{2.16}$$

(d) *There exist C_1 and C_2 such that*

$$(C_1 u\chi(a_u))^{-1} \leqslant a_u'/a_u \leqslant (C_2 u\chi(a_u/2))^{-1}, \qquad u \in [0, \infty). \tag{2.17}$$

(e) *There exists C such that*

$$a_{ru}/a_u \geqslant 1 + C(\log r)/\chi(a_{ru}), \qquad r \in [1, \infty), \ u \in (0, \infty). \tag{2.18}$$

(f) *For each fixed* $L > 0$,

$$\lim_{u \to \infty} a_{Lu}/a_u = 1. \tag{2.19}$$

(g) *For each fixed* $\delta > 0$,

$$\lim_{n \to \infty} a_n n^{-\delta} = 0. \tag{2.20}$$

Proof. (a) From (1.7),

$$\frac{n}{a_n Q'(a_n)} = \frac{2}{\pi} \int_0^1 \frac{tQ'(a_n t)}{Q'(a_n)} \frac{dt}{(1-t^2)^{1/2}}. \tag{2.21}$$

By Lemma 2.1(e) (with $j=1$), the integrand in this last integral has limit 0 as $n \to \infty$, for each fixed $t \in (0, 1)$. Further, as $sQ'(s)$ is increasing in $(0, \infty)$, we see that the integrand is bounded above by $(1-t^2)^{-1/2}$, for $n \geqslant 1$, $t \in (0, 1)$. Then Lebesgue's Dominated Convergence Theorem yields

$$\lim_{n \to \infty} n/(a_n Q'(a_n)) = 0,$$

and (2.14) is true for $j=1$. For $j=2$, we use (see (2.13))

$$a_n^2 Q''(a_n)/n = \{a_n Q'(a_n)/n\}\{\chi(a_n) - 1\}, \tag{2.22}$$

as well as (2.4) and (2.14) for $j=1$.

It remains to prove (2.14) for $j=0$. Now if $0 < \delta < \frac{1}{2}$, (1.7) yields

$$\begin{aligned} n/Q(a_n) &\geqslant \frac{2}{\pi} \int_{1-\delta}^1 \frac{a_n t Q'(a_n t)}{Q(a_n)(1-t^2)^{1/2}} dt \\ &\geqslant \frac{2}{\pi} \frac{(1-\delta)[Q(a_n) - Q(a_n(1-\delta))]}{Q(a_n)(1-(1-\delta)^2)^{1/2}} \\ &\geqslant \frac{2}{\pi} \frac{(1-\delta)[Q(a_n)/2]}{Q(a_n)(2\delta)^{1/2}}, \end{aligned}$$

for n large enough, by Lemma 2.1(e). Since δ may be made arbitrarily small, (2.14) follows for $j=0$.

(b) For $j=0$, the monotonicity of Q and (a) yield (2.15), even uniformly for $x \in [-1, 1]$. To prove (2.15) for $j=1$, let $0 < \delta < \frac{1}{3}$, and $\delta \leqslant |x| \leqslant 1 - 2\delta$. For $n \geqslant n_0(\delta)$,

$$\frac{Q(a_n)}{a_nQ'(a_nx)} \geqslant \frac{Q(a_n)-Q(a_n(1-\delta))}{a_nQ'(a_n(1-2\delta))}$$
$$= \frac{\int_{a_n(1-\delta)}^{a_n} Q'(u)\,du}{a_nQ'(a_n(1-2\delta))}$$
$$\geqslant \frac{\delta Q'(a_n(1-\delta))}{Q'(a_n(1-2\delta))} \to \infty \qquad \text{as} \quad n \to \infty,$$

by Lemma 2.1(e). Then as $Q(a_n)=o(n)$, (2.15) follows for $j=1$. For $j=2$, one similarly estimates $Q'(a_n(1-\delta))/\{a_nQ''(a_nx)\}$.

(c) Let

$$r := r(n) := 1-\chi(a_n)^{-1}.$$

We have from (2.21) and Lemma 2.1(c) that

$$\frac{n}{a_nQ'(a_n)} \geqslant \frac{2}{\pi}\int_0^1 t^{C\chi(a_n)}(1-t^2)^{-1/2}\,dt$$
$$\geqslant \frac{2}{\pi} r^{C\chi(a_n)} \int_r^1 (1-t^2)^{-1/2}\,dt$$
$$\geqslant C_1\chi(a_n)^{-1/2},$$

by choice of r. So (2.16) is valid for $j=1$. Then for $j=2$, (2.22) yields (2.16).

(d) From (1.7), we deduce that for $u \in (0, \infty)$,

$$1 = \frac{a'_u}{a_u}\frac{2}{\pi}\int_0^1 a_utQ'(a_ut)\chi(a_ut)(1-t^2)^{-1/2}\,dt.$$

Since χ is quasi-increasing in $(0, \infty)$, we have from (1.7),

$$1 \leqslant C_1\frac{a'_u}{a_u}\chi(a_u)u.$$

In the other direction, we have

$$1 \geqslant C_2\frac{a'_u}{a_u}\chi(a_u/2)\int_{1/2}^1 a_utQ'(a_ut)(1-t^2)^{-1/2}\,dt$$
$$\geqslant C_2\frac{a'_u}{a_u}\chi(a_u/2)\,u/2$$

since $a_utQ'(a_ut)(1-t^2)^{-1/2}$ is an increasing function of $t \in (0, 1)$.

(e) For $r > 1$ and $u \in (0, \infty)$,

$$\begin{aligned} a_{ru}/a_u &= \exp\left(\int_u^{ru} a_t'/a_t \, dt\right) \\ &\geqslant \exp\left(C_1 \int_u^{ru} (\chi(a_t)t)^{-1} \, dt\right) \\ &\geqslant \exp(C_2 \chi(a_{ru})^{-1} \log r) \\ &\geqslant 1 + C_2 \chi(a_{ru})^{-1} \log r. \end{aligned}$$

(f) It suffices to consider the case $L > 1$. Now by (d) of this lemma,

$$\begin{aligned} a_{Lu}/a_u &= \exp\left(\int_u^{Lu} a_t'/a_t \, dt\right) \\ &\leqslant \exp\left(\int_u^{Lu} (C_2 t \chi(a_t/2))^{-1} \, dt\right) \\ &\leqslant \exp(C_2 \chi(a_u/2)^{-1} \log L) \to 1 \qquad \text{as} \quad u \to \infty. \end{aligned}$$

(g) We see that

$$\frac{d}{du} \{a_u/u^{\delta/2}\} = \{a_u/u^{\delta/2}\}\{a_u'/a_u - \delta/(2u)\}.$$

Then Lemma 2.2(d) shows that for large enough u, this last right-hand side is negative, and so $a_u/u^{\delta/2}$ is a decreasing positive function of u, for large enough u. Then (2.20) follows. ∎

Finally, one more lemma on a_n:

Lemma 2.3. *Let $W(x)$ be as in Lemma 2.1, satisfying in addition, for some $0 < \eta < 1$,*

$$\chi(x) = O(Q'(x)^{2\eta}), \qquad x \to \infty. \tag{2.23}$$

(a) *Then as $n \to \infty$,*

$$Q'(a_n) = O((n/a_n)^{1/(1-\eta)}), \tag{2.24}$$

$$\chi(a_n) = O((n/a_n)^{2\eta/(1-\eta)}), \tag{2.25}$$

and

$$a_n Q''(a_n) = O((n/a_n)^{(2\eta+1)/(1-\eta)}). \tag{2.26}$$

(b) *Suppose*

$$m = m(n) = n[1 + O((n/a_n)^{-2\eta/(1-\eta)})], \qquad n \to \infty. \tag{2.27}$$

Then

$$\lim_{n \to \infty} Q'(a_m)/Q'(a_n) = 1. \tag{2.28}$$

(c) *Suppose*

$$x = x(n) = a_n[1 + o((n/a_n)^{-2\eta/(1-\eta)})], \qquad n \to \infty, \tag{2.29}$$

Then as $n \to \infty$,

$$Q'(x) = O((n/a_n)^{1/(1-\eta)}), \tag{2.30}$$

and

$$a_n Q''(x) = O((n/a_n)^{(2\eta-1)/(1-\eta)}). \tag{2.31}$$

Proof. (a) From (2.16) for $j = 1$,

$$a_n Q'(a_n)/n = O(\chi(a_n)^{1/2}) = O(Q'(a_n)^{\eta}),$$

so

$$Q'(a_n)^{1-\eta} = O(n/a_n).$$

Then (2.24) follows, while (2.23) yields (2.25). Finally, (2.22) yields (2.26).

(b) We have if $m = m(n) \geqslant n$, for n large enough,

$$\begin{aligned}
1 &\leqslant Q'(a_m)/Q'(a_n) \\
&= \exp\left(\int_n^m \{Q''(a_t)/Q'(a_t)\}\, a_t'\, dt\right) \\
&= \exp\left(\int_n^m (\chi(a_t) - 1)\, a_t'/a_t\, dt\right) \\
&\leqslant \exp(C_2[\chi(a_m)/\chi(a_n/2)] \log(m/n)) \\
&\qquad \text{(by Lemma2.2(d))} \\
&\leqslant \exp(O((m/a_m)^{2\eta/(1-\eta)})\, o(1)\, O((a_n/n)^{2\eta/(1-\eta)})) \to 1 \\
&\qquad \text{as } n \to \infty,
\end{aligned}$$

since $m \sim n$ as $n \to \infty$. Similarly, we may handle the case $m \leqslant n$.

(c) We have from (2.25) and then from Lemma 2.2(e) that

$$x = a_n\{1 + o(\chi(a_{2n})^{-1})\} \leqslant a_{2n},$$

n large enough. Then the monotonicity of Q'' and Q' and (2.24) and (2.26) yield (2.30)–(2.31). ▮

3. Majorization of Weighted Polynomials and Estimation of $U_n(t)$

Following is a summary of the results that we need on the majorization of weighted polynomials.

Lemma 3.1. *Let $W(x) := e^{-Q(x)}$ be as in Lemma 2.1. Assume in addition that for some $1 < p < 2$,*

$$\|Q'\|_{L_p[0,1]} < \infty. \tag{3.1}$$

(a) *For $n = 1, 2, 3, \ldots,$ and $x \in (-1, 1)$, let*

$$\mu_n(x) := \frac{2}{\pi^2} \int_0^1 \frac{(1-x^2)^{1/2}}{(1-s^2)^{1/2}} \frac{a_n s Q'(a_n s) - a_n x Q'(a_n x)}{n(s^2 - x^2)}\, ds. \tag{3.2}$$

Then $\mu_n(x)$ is even, finite a.e. in $(-1, 1)$,

$$\mu_n(x) \geqslant 0 \qquad \text{a.e. in } (-1, 1), \tag{3.3}$$

$$\int_{-1}^{1} \mu_n(x)\, dx = 1, \tag{3.4}$$

and, with p as above,

$$\|\mu_n\|_{L_p[-1,1]} \leqslant C \|Q'(a_n t)(1-t^2)^{-1/2}\|_{L_p[-1,1]} (a_n/n). \tag{3.5}$$

(b) *For $n = 1, 2, 3, \ldots,$ let*

$$A_n := \frac{2}{n\pi^2} \int_0^1 \frac{a_n Q'(a_n) - a_n t Q'(a_n t)}{(1-t^2)^{3/2}}\, dt. \tag{3.6}$$

Then, if $'$ denotes differentiation with respect to t,

$$A_n = \frac{2}{n\pi^2} \int_0^1 \frac{t(a_n t Q'(a_n t))'}{(1-t^2)^{1/2}}\, dt. \tag{3.7}$$

There exist C_1 and C_2 such that

$$C_1 \chi(a_n/2) \leqslant A_n \leqslant C_2 \chi(a_n). \tag{3.8}$$

Further, there exists C such that for $x \in [\frac{7}{8}, 1]$ *and* $n = 1, 2, 3, \ldots,$

$$|\mu_n(x)(1-x^2)^{-1/2} - A_n| \leqslant C\chi(a_n)^{3/2}\,(1-x)^{1/5}. \tag{3.9}$$

Finally,

$$\int_{-1}^{1} \mu_n(x)/(1-x)\,dx = a_n Q'(a_n)/n. \tag{3.10}$$

(c) *For* $n = 1, 2, 3, \ldots,$ *and* $z \in \mathbb{C}$, *let*

$$U_n(z) := \int_{-1}^{1} \log|z-t|\,\mu_n(t)\,dt - Q(a_n|z|)/n + \chi_n/n, \tag{3.11}$$

where

$$\chi_n := 2\pi^{-1} \int_0^1 \frac{Q(a_n t)}{(1-t^2)^{1/2}}\,dt + n\log 2. \tag{3.12}$$

Then

$$U_n(x) = 0, \qquad x \in [-1, 1], \tag{3.13}$$

and there exists $C > 0$ *such that as* $\varepsilon \to 0+$,

$$\begin{aligned} U_n'(1+\varepsilon) = {} & -A_n\pi(2\varepsilon)^{1/2} + O(\varepsilon^{2/3}\chi(a_n)^{3/2}) \\ & + O[\varepsilon\chi(a_n(1+\varepsilon))^{3/2}\,(1+\varepsilon)^{C\chi(a_n(1+\varepsilon))}], \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} U_n(1+\varepsilon) = {} & -A_n\pi\sqrt{8}\,\varepsilon^{3/2}/3 + O(\varepsilon^{5/3}\chi(a_n)^{3/2}) \\ & + O[\varepsilon^2\chi(a_n(1+\varepsilon))^{3/2}\,(1+\varepsilon)^{C\chi(a_n(1+\varepsilon))}]. \end{aligned} \tag{3.15}$$

Further,

$$U_n^{(j)}(x) < 0, \qquad x \in (1, \infty),\ j = 0, 1, \tag{3.16}$$

and

$$(xU_n'(x))' < 0, \qquad x \in (1, \infty). \tag{3.17}$$

(d) *For* $n = 1, 2, 3, \ldots,$ $P \in \mathscr{P}_n$, *and* $z \in \mathbb{C} \setminus [-1, 1]$,

$$|P(a_n z)\,W(a_n|z|)| \leqslant \|PW\|_{[-a_n, a_n]} \exp(nU_n(z)). \tag{3.18}$$

Furthermore,

$$\|PW\|_{\mathbb{R}} = \|PW\|_{[-a_n, a_n]}, \tag{3.19}$$

and if P is not identically zero,

$$|PW|(x) < \|PW\|_{\mathbb{R}}, \qquad |x| > a_n. \tag{3.20}$$

Proof. (a) First, (3.3), (3.4), and (3.5) follow from (a) of Lemma 5.3 in [16] with $R := a_n$, $\mu_n := \mu_{n,a_n}$ and so on. Note that $B_{n,a_n} = 0$ (see (5.44) in [16, p. 37]).

(b) First, (3.7) follows from (3.6) by an integration by parts (see (5.57) in [16]). Next, we see that

$$\begin{aligned} A_n &= \frac{2}{n\pi^2} \int_0^1 \frac{a_n t Q'(a_n t)}{(1-t^2)^{1/2}} \chi(a_n t)\, dt \\ &\leqslant C\chi(a_n), \end{aligned}$$

as χ is quasi-increasing, and by the definition (1.7) of a_n. For the lower bound, we have

$$A_n \geqslant C\chi(a_n/2) \frac{2}{n\pi^2} \int_{1/2}^1 \frac{a_n t Q'(a_n t)}{(1-t^2)^{1/2}}\, dt \geqslant C\chi(a_n/2)(1/(2\pi)),$$

as $sQ'(s)$ is increasing in $(0, \infty)$, and by (1.7). This yields (3.8).

To prove (3.9), we note from (5.49) in [16] that (3.9) is true, but with the right-hand side of (3.9) replaced by $C_3(1-x)^{1/5}\, \tau_n$, where

$$\begin{aligned} \tau_n &:= a_n Q'(a_n)/n + \max\{|a_n^2 Q''(a_n u)|/n \colon u \in [\tfrac{1}{2}, 1]\} \\ &\leqslant C\chi(a_n)^{3/2}, \end{aligned} \tag{3.21}$$

by (2.16) with $j = 1, 2$, and since $Q''(x)$ and $\chi(x)$ are increasing for large x (see Lemma 2.1(b)). Then (3.9) follows. Finally, (3.10) is a restatement of (5.50) in [16, p. 40].

(c) First, (3.13) follows from (5.45) in [16]. Next, (3.14) was shown to be true in [16, (5.53)], but with the order terms in (3.14) replaced by

$$O(\varepsilon^{2/3}\tau_n) + O(\varepsilon\rho_{n,\varepsilon}), \tag{3.22}$$

where τ_n is as at (3.21) and where

$$\begin{aligned} \rho_{n,\varepsilon} &:= \max\{a_n^2 |Q''(a_n u)|/n \colon u \in [1, 1+\varepsilon]\}, \\ &\leqslant a_n^2 Q''(a_n(1+\varepsilon))/n, \end{aligned} \tag{3.23}$$

for n large enough, since $Q''(x)$ is increasing for large x. Now, using (2.6),

$$\begin{aligned} &a_n^2 Q''(a_n(1+\varepsilon))/n \\ &\quad = (1+\varepsilon)^{-2}\{\chi(a_n(1+\varepsilon))-1\}\, a_n(1+\varepsilon)\, Q'(a_n(1+\varepsilon))/n \\ &\quad \leqslant C_1\chi(a_n(1+\varepsilon))(1+\varepsilon)^{C\chi(a_n(1+\varepsilon))}\, a_n Q'(a_n)/n \\ &\quad \leqslant C_2\chi(a_n(1+\varepsilon))^{3/2}\,(1+\varepsilon)^{C\chi(a_n(1+\varepsilon))}, \end{aligned}$$

by (2.16). Then using (3.21), we obtain

$$\begin{aligned} &O(\varepsilon^{2/3}\tau_n) + O(\varepsilon\rho_{n,\varepsilon}) \\ &\quad \leqslant C_1[\varepsilon^{2/3}\chi(a_n)^{3/2} + \varepsilon\chi(a_n(1+\varepsilon))^{3/2}\,(1+\varepsilon)^{C\chi(a_n(1+\varepsilon))}], \end{aligned}$$

and (3.14) follows as stated. Next, integrating (3.14) yields (3.15). Finally, (3.16) and (3.17) follow from (5.55) to (5.56) in [16] with $R = a_n$.

(d) This follows from Theorem 7.1(i), (ii) in [16, pp. 49–50]. ∎

We next need to derive some estimates for $\mu_n(t)$:

LEMMA 3.2. *Let $W(x)$ be as in Lemma 2.1, with the additional restriction that $Q''(x)$ is continuous in $\mathbb{R}$. Let $\xi > 0$ and for n large enough, let $\psi_n(x)$ and A_n^* be given by (1.23) and (1.24), respectively. Then*

(a) *Given $0 < \varepsilon < 1$, we have for n large enough,*

$$\mu_n(x) \sim 1, \qquad \text{uniformly for} \quad 0 \leqslant x \leqslant 1-\varepsilon. \tag{3.24}$$

(b) *There exist C_1 and C_2 such that for n large enough, and uniformly for $C_1/a_n \leqslant x \leqslant 1$,*

$$\psi_n(x) \geqslant C_2(1-x)^{1/2}\{a_n x Q''(a_n x) + Q'(a_n x)\} + C_3 x Q'(a_n x). \tag{3.25}$$

(c) *Given $0 < \varepsilon < 1$, we have for n large enough,*

$$\mu_n(x) \sim (1-|x|)^{1/2}\, a_n \psi_n(|x|)/n, \qquad \text{uniformly for} \quad \varepsilon \leqslant |x| < 1. \tag{3.26}$$

(d) *Given $0 < \varepsilon < 1$, we have for n large enough,*

$$\psi_n(x) \sim n/a_n, \qquad \text{uniformly for} \quad 0 \leqslant x \leqslant 1-\varepsilon. \tag{3.27}$$

(e) *For n large enough, $\psi_n(t)$ is quasi-increasing in $(0, 1)$, with the constant in the definition of quasi-increasing functions being independent of n.*

(f) *Let A_n be defined by (3.6). Then for n large enough,*

$$A_n^* \sim A_n = O(\chi(a_n)), \qquad n \to \infty. \tag{3.28}$$

(g) *If $r \in (0, \infty)$, then we have for n large enough,*

$$\psi_n(x) \sim nA_n^*/a_n, \tag{3.29}$$

uniformly for

$$1 \geq x \geq 1 - r\chi(a_n)^{-1/2}. \tag{3.30}$$

(h) *There exists C such that*

$$\mu_n(x) \leq C\{a_n Q'(a_n)/n\}, \qquad x \in [0, 1],\ n \geq 1. \tag{3.31}$$

Proof. We note first that there exists κ such that $(xQ'(x))' = \chi(x)Q'(x)$ is increasing for $x \in [\kappa, \infty)$, that is, $xQ'(x)$ is convex in $[\kappa, \infty)$. It then follows that for each fixed $v \in [\kappa, \infty)$,

$$\frac{uQ'(u) - vQ'(v)}{u - v}$$

is an increasing positive function of $u \in [\kappa, \infty)$. It is also positive for $u, v \in (0, \infty)$, by (2.2). We assume that $\kappa \geq \xi$ below. Further, note that the continuity of Q'', and hence of Q', ensures that (3.1) is true for any $p > 1$.

(a) Let $0 < \varepsilon < \frac{1}{2}$. Since $\mu_n(\cdot)$ is even, it suffices to consider $x \in [0, 1 - 2\varepsilon]$. We have from (3.2) that

$$\begin{aligned}
\mu_n(x) \leq{} & \frac{2}{\pi^2}(1 - (1-\varepsilon)^2)^{-1/2} \\
& \times \frac{a_n}{n}\int_0^{1-\varepsilon} \frac{a_n sQ'(a_n s) - a_n xQ'(a_n x)}{a_n s - a_n x}\,\frac{ds}{s + x} \\
& + \frac{2}{\pi^2}\int_{1-\varepsilon}^{1} (1 - s^2)^{-1/2}\,\frac{a_n sQ'(a_n s) - a_n xQ'(a_n x)}{n(s^2 - (1 - 2\varepsilon)^2)}\,ds \\
\leq{} & C\left\{\frac{a_n}{n}\int_0^{1-\varepsilon} (vQ'(v))'\,\frac{ds}{s + x}\right. \\
& \left. + n^{-1}\int_{1-\varepsilon}^{1} (1 - s^2)^{-1/2}\,a_n sQ'(a_n s)\,ds\right\},
\end{aligned}$$

where v lies between $a_n s$ and $a_n x$, and we have used the properties of $Q'(t)$ in $(0, \infty)$. Here

$$\begin{aligned}
(vQ'(v))'/(s + x) &= a_n\chi(v)\,Q'(v)/(a_n s + a_n x) \\
&\leq a_n\chi(v)Q'(v)/v \\
&\leq C_1 a_n\chi(a_n(1-\varepsilon))\,Q'(a_n(1-\varepsilon))/(a_n(1-\varepsilon)),
\end{aligned}$$

since $\chi(\cdot)$ is quasi-increasing, and by (2.11) of Lemma 2.1(h). Then

$$\frac{a_n}{n}(vQ'(v))'/(s+x)$$
$$\leqslant C_2\left\{a_nQ'(a_n(1-\varepsilon))+a_n^2Q''(a_n(1-\varepsilon))\right\}\Big/n=o(1),$$

as $n\to\infty$, by (2.13) and Lemma 2.2(b). Then, using (1.7), we obtain

$$\mu_n(x)\leqslant C\{o(1)+C_2\},$$

uniformly for $|x|\leqslant 1-2\varepsilon$, and n large enough. In the other direction, we have for $|x|\leqslant 1-2\varepsilon$ that

$$\mu_n(x)\geqslant\frac{2}{\pi^2}(1-(1-2\varepsilon)^2)^{1/2}$$
$$\times\int_{1-\varepsilon}^{1}(1-s^2)^{-1/2}\frac{a_nsQ'(a_ns)-a_n(1-2\varepsilon)\,Q'(a_n(1-2\varepsilon))}{ns^2}\,ds$$
$$\geqslant Cn^{-1}\int_{1-\varepsilon}^{1}(1-s^2)^{-1/2}\,a_nsQ'(a_ns)\,ds,$$

using Lemma 2.1(e). Finally, (1.7) and Lemma 2.1(e) with $j=1$ yield for n large enough that

$$\mu_n(x)\geqslant C_1,\qquad |x|\leqslant 1-2\varepsilon.$$

(b) The comment at the beginning of the proof shows that

$$\frac{a_nxQ'(a_nx)-a_nsQ'(a_ns)}{a_nx-a_ns}$$

is an increasing function of $x\in[\kappa/a_n,\infty)$ for each fixed $s\in[\kappa/a_n,\infty)$ and takes the value $(vQ'(v))'|_{v=a_nx}$ when $s=x$. It is also positive for all $x, s>0$, by (2.2). Then for $x\in[\kappa/a_n, 1)$,

$$\psi_n(x)\geqslant\int_x^1(1-s)^{-1/2}(vQ'(v))'|_{v=a_nx}\,ds$$
$$\geqslant C(1-x)^{1/2}\{a_nxQ''(a_nx)+Q'(a_nx)\},$$

which is part of the lower bound in (3.25). Next, if $1\geqslant x\geqslant 4\xi/a_n$, (1.23) shows that

$$\psi_n(x) \geqslant \int_{x/4}^{x/2} (1-s)^{-1/2} \frac{a_n x Q'(a_n x) - a_n s Q'(a_n s)}{a_n x - a_n s}\, ds$$

$$\geqslant (x/4) \frac{a_n x Q'(a_n x) - a_n(x/2)\, Q'(a_n x/2)}{a_n x}$$

$$\geqslant (4a_n)^{-1}\, a_n x Q'(a_n x)\{1 - 2^{-\chi(a_n x/2)/C}\} \geqslant C_4 x Q'(a_n x),$$

by Lemma 2.1(c) and the fact that $\chi(\cdot)$ is bounded below by a positive number in $(0, \infty)$. This completes the proof of (3.25).

(c) It suffices to consider $x \in [\varepsilon, 1)$. Note first that

$$(1-t^2)^{1/2} \sim (1-t)^{1/2}, \qquad t \in [0, 1),$$

and

$$(s+x)^{-1} \sim 1,$$

uniformly for $x \geqslant \varepsilon$, and $s \in [0, 1]$. Next, for n large enough, and for $x \geqslant \varepsilon$,

$$0 \leqslant I(n, x) := \int_0^{\xi/a_n} \frac{(1-x^2)^{1/2}}{(1-s^2)^{1/2}} \frac{a_n s Q'(a_n s) - a_n x Q'(a_n x)}{n(s^2 - x^2)}\, ds$$

$$\leqslant C_1(1-x)^{1/2}\, (\xi/a_n)(a_n x Q'(a_n x))/n$$

$$\leqslant C_2 a_n^{-1}(1-x)^{1/2}\, a_n \psi_n(x)/n, \tag{3.32}$$

by (b) of this lemma. These remarks, and the definitions (1.23) of ψ_n and (3.2) of μ_n, easily yield (3.26).

(d) The proof of this is very similar to that of (a).

(e) Recalling that $\xi \leqslant \kappa$, suppose first that $\xi = \kappa$. Then the remarks at the beginning of the lemma even show that $\psi_n(x)$ is increasing in $(\xi/a_n, 1)$. For $x \in (0, \xi/a_n]$, we use (d) of this lemma to show that $\psi_n(x)$ is quasi-increasing, uniformly in n. When $\xi < \kappa$, one can split the integral defining ψ_n into integrals from ξ/a_n to κ/a_n, and from κ/a_n to 1. The second integral may be treated by the argument for the case $\xi = \kappa$. The first integral may be shown to be much smaller than the second integral, by estimations similar to that at (3.32) and by continuity of Q'' near 0.

(f) From (3.7) and (1.7),

$$A_n = \frac{2}{n\pi^2} \int_0^1 \frac{a_n t Q'(a_n t) + (a_n t)^2\, Q''(a_n t)}{(1-t^2)^{1/2}}\, dt$$

$$\begin{cases} \leqslant \pi^{-1} + J, \\ \geqslant J, \end{cases} \tag{3.33}$$

where

$$J := \frac{2}{n\pi^2} \int_0^1 \frac{(a_n t)^2 Q''(a_n t)}{(1-t^2)^{1/2}} \, dt.$$

Since uniformly for $t \in [0, \frac{1}{2}]$ (recall now Q'' is continuous at 0, and recall Lemma 2.2(b)),

$$\lim_{n \to \infty} (a_n t)^2 Q''(a_n t)/n = 0,$$

the result follows from the definition (1.24) of A_n^*, and from (3.8), which shows that

$$\lim_{n \to \infty} A_n = \infty.$$

(g) From (3.26) and (3.9), for $x \in [\frac{7}{8}, 1]$, and $n = 1, 2, 3, \ldots$,

$$\begin{aligned} \psi_n(x) &\sim (n/a_n)\, \mu_n(x)(1-x^2)^{-1/2} \\ &= (n/a_n)\{A_n + O[\chi(a_n)^{3/2}\,(1-x)^{1/5}]\} \\ &= (nA_n/a_n)\{1 + o[\chi(a_n)^{3/2}\,(1-x)^{1/5}]\}. \end{aligned}$$

Then for the range (3.30), we obtain (3.29), usig (3.28).

(h) Since (see Lemma 2.2(a))

$$\lim_{n \to \infty} a_n Q'(a_n)/n = \infty,$$

Lemma 3.2(a) implies the bound (3.31) for $|x| \leqslant \frac{1}{2}$, and n large enough. Next, by (c) and (e) of this lemma, for $\frac{1}{2} \leqslant x \leqslant 1$, and n large enough,

$$\begin{aligned} \mu_n(x) &\sim (1-x)^{1/2}\,(a_n/n)\, \psi_n(x) \\ &\leqslant C(1-x)^{-1/2}\,(a_n/n) \int_x^1 \psi_n(s)\, ds \\ &\leqslant C \int_x^1 (a_n/n)(1-s)^{-1/2}\, \psi_n(s)\, ds. \end{aligned}$$

Using (c) again, we obtain

$$\mu_n(x) \leqslant C_1 \int_x^1 \frac{\mu_n(s)}{1-s}\, ds \leqslant C_1 a_n Q'(a_n)/n,$$

by (3.10). ∎

We proceed to estimate $U_n(t)$ for t near $[-1, 1]$.

LEMMA 3.3. *Let $W(x)$ be as in Lemma 2.1, with the additional restriction that Q'' is continuous in $\mathbb{R}$.*

(a) *For $x, y \in \mathbb{R}$ and $n \geqslant 1$,*

$$U_n(x+iy) \leqslant \int_0^1 \log[1+(y/(|x|-t))^2]\, \mu_n(t)\, dt. \tag{3.34}$$

(b) *Let $0<\varepsilon<1$. For $|x| \leqslant 1-\varepsilon$, $|y| \leqslant 1$, and $n \geqslant 1$,*

$$U_n(x+iy) \leqslant C|y|. \tag{3.35}$$

(c) *For $x \in \mathbb{R}$, $|y| \leqslant 1$, and $n \geqslant 1$,*

$$U_n(x+iy) \leqslant C\{a_n Q'(a_n)/n\}\, |y|. \tag{3.36}$$

Proof. (a) From (3.13) and (3.16), we have

$$\begin{aligned} U_n(x+iy) &\leqslant U_n(x+iy) - U_n(x) \\ &= \int_{-1}^1 \log|x+iy-t|\, \mu_n(t)\, dt - \int_{-1}^1 \log|x-t|\, \mu_n(t)\, dt \\ &\quad - Q(a_n(x^2+y^2)^{1/2})/n + Q(a_n|x|)/n \qquad \text{(by (3.11))} \\ &\leqslant \frac{1}{2}\int_{-1}^1 \log\{1+(y/(x-t))^2\}\, \mu_n(t)\, dt, \end{aligned}$$

as $Q(\cdot)$ is increasing in $(0, \infty)$. Since $\mu_n(t)$ is even and

$$|y|/(x+t) \leqslant |y|/(x-t), \qquad x, t \in [0, 1],$$

we obtain (3.34) for $x \in [0, \infty)$ and $y \in \mathbb{R}$. The fact that $U_n(-x+iy) = U_n(x+iy)$ yields the result for $x \in \mathbb{R}$.

(b) From (a) above, and from Lemma 3.2(a), we have for $|x| \leqslant 1-\varepsilon$ that

$$\begin{aligned} U_n(x+iy) &\leqslant C \int_0^{1-\varepsilon/2} \log\{1+(y/(|x|-t))^2\}\, dt \\ &\quad + \int_{1-\varepsilon/2}^1 \log\{1+(y/(\varepsilon/2))^2\}\, \mu_n(t)\, dt \\ &\leqslant C|y| \int_{(|x|-1-\varepsilon/2)/|y|}^{|x|/|y|} \log(1+u^{-2})\, du + (2y/\varepsilon)^2 \int_0^1 \mu_n(t)\, dt, \end{aligned}$$

by the substitution $t = |x| - u|y|$ in the first integral, and using the inequality

$$\log(1+s) \leqslant s, \qquad s \in (0, \infty), \tag{3.37}$$

in the second integral. As $|y| \leqslant 1$, we obtain

$$U_n(x+iy) \leqslant C|y| \int_{-\infty}^{\infty} \log(1+u^{-2})\, du + (2/\varepsilon)^2\, |y|.$$

(c) By Lemma 3.2(h), and (a) above,

$$U_n(x+iy) \leqslant C\{a_n Q'(a_n)/n\} \int_0^1 \log\{1+(y/(|x|-t))^2\}\, dt.$$

Then, making the substitution $t=|x|-u|y|$, we obtain (3.36), much as before. ∎

We need a better estimate for $|x|$ close to 1:

LEMMA 3.4. *Let $W(x)$ be as in Lemma 2.1, with the additional restriction that $Q''(x)$ is continuous in $\mathbb{R}$.*

(a) *Let $0<\eta<1$. There exist C_1 and C_2 such that for $\eta \leqslant |x| < 1$, $|y| \leqslant 1$, and $n \geqslant C_1$,*

$$U_n(x+iy) \leqslant C_2 y^2 + C_2 \left[\frac{|y|}{\delta(x)} \int_{|x|+\delta(|x|)}^1 \mu_n(t)\, dt\right] \times [1+(|y|/\delta(x))^{1/2}], \tag{3.38}$$

where

$$\delta(x) := (1-|x|)/2. \tag{3.39}$$

(b) *There exist C_1, C_2, and C_3 such that for $|x| \in [1, \infty)$, $|y| \leqslant 1$, and $n \geqslant C_1$,*

$$U_n(x+iy) \leqslant C_2 A_n^* y^{3/2} \leqslant C_3 \chi(a_n)\, y^{3/2}. \tag{3.40}$$

Proof. Note first that $|x|+\delta(x) = (1+|x|)/2 < 1$ for $|x|<1$, while

$$1-(|x|+\delta(x)) = \delta(x).$$

(a) From Lemma 3.2(c), and Lemma 3.3(a) for $\eta \leqslant |x| < 1$,

$$\begin{aligned} U_n(x+iy) \leqslant & \int_0^{\eta/2} \log[1+(y/(\eta/2))^2]\, \mu_n(t)\, dt \\ & + C_3 \int_{\eta/2}^{|x|-\delta(x)} \log[1+(y/(|x|-t))^2]\, \frac{a_n}{n} (1-t)^{1/2}\, \psi_n(t)\, dt \\ & + \int_{|x|+\delta(x)}^1 \log[1+(y/\delta(x))^2]\, \mu_n(t)\, dt \\ =: & \; T_1 + T_2 + T_3, \end{aligned} \tag{3.41}$$

say. Here, using the inequality (3.37), we obtain

$$T_1 \leqslant 4y^2/\eta^2 \int_{-1}^{1} \mu_n(t)\,dt = 4y^2/\eta^2. \tag{3.42}$$

Next, using the fact that ψ_n is quasi-increasing, we obtain

$$\begin{aligned} T_2 &\leqslant C(a_n/n)\,\psi_n(|x|+\delta(x)) \\ &\quad\times \int_{\eta/2}^{|x|-\delta(x)} \log[1+(y/(|x|-t))^2]\,(1-t)^{1/2}\,dt \\ &= C(a_n/n)\,\psi_n(|x|+\delta(x))\,|y| \\ &\quad\times \int_{(\eta/2-|x|)/|y|}^{\delta(x)/|y|} \log(1+u^{-2})(1-|x|-u|y|)^{1/2}\,du, \end{aligned}$$

by the substitution $t=|x|+u|y|$. Using the inequality

$$(a+b)^{1/2} \leqslant |a|^{1/2}+|b|^{1/2}, \qquad a, b\in\mathbb{R}, \text{ such that } a+b\geqslant 0,$$

we obtain

$$\begin{aligned} T_2 &\leqslant C(a_n/n)\,\psi_n(|x|+\delta(x))\,|y| \\ &\quad\times\left\{(2\delta(x))^{1/2}\int_{-\infty}^{\infty}\log(1+u^{-2})\,du+|y|^{1/2}\int_{-\infty}^{\infty}|u|^{1/2}\log(1+u^{-2})\,du\right\} \\ &\leqslant C(a_n/n)\,\psi_n(|x|+\delta(x))\,|y|\,\delta(x)^{1/2}\,\{1+C(|y|/\delta(x))^{1/2}\}. \end{aligned}$$

Next,

$$\begin{aligned} &(a_n/n)\,\psi_n(|x|+\delta(x))\,\delta(x)^{1/2} \\ &\quad\leqslant C_2(a_n/n)\,\psi_n(|x|+\delta(x))\,\delta(x)^{-1}\int_{|x|+\delta(x)}^{1}(1-t)^{1/2}\,dt \\ &\quad\leqslant C_3\,\delta(x)^{-1}\int_{|x|+\delta(x)}^{1}(a_n/n)\,\psi_n(t)(1-t)^{1/2}\,dt \\ &\quad\leqslant C_4\,\delta(x)^{-1}\int_{|x|-\delta(x)}^{1}\mu_n(t)\,dt, \end{aligned}$$

by Lemma 3.2(c). Hence

$$T_2 \leqslant C_5(|y|/\delta(x))\int_{|x|+\delta(x)}^{1}\mu_n(t)\,dt\{1+(|y|/\delta(x))^{1/2}\}. \tag{3.43}$$

Finally, we see from (3.37) that

$$T_3 \leqslant \log[1+(|y|/\delta(x))]^2 \int_{|x|+\delta(x)}^{1} \mu_n(t)\,dt$$

$$\leqslant 2(|y|/\delta(x)) \int_{|x|+\delta(x)}^{1} \mu_n(t)\,dt. \tag{3.44}$$

Combining (3.41) to (3.44) yields (3.38).

(b) Since the constants in (3.38) are independent of n and x, and since the left-hand side is continuous at ± 1, we may let $|x| \to 1$, to deduce that for $|y| \leqslant 1$, $n \geqslant C_1$,

$$U_n(\pm 1+iy) \leqslant C_2 y^2 + C_2|y| \left\{ \limsup_{x \to 1-} \delta(x)^{-1} \int_{|x|+\delta(x)}^{1} \mu_n(t)\,dt \right.$$
$$\left. + |y|^{1/2} \limsup_{x \to 1-} \delta(x)^{-3/2} \int_{|x|-\delta(x)}^{1} \mu_n(t)\,dt \right\}.$$

Using Lemma 3.2(c) and (g), we easily obtain for $|y| \leqslant 1$, $n \geqslant C_1$ that

$$U_n(\pm 1+iy) \leqslant C_2 y^2 + C_2|y|^{3/2} A_n^* \leqslant C_3|y|^{3/2} A_n^*. \tag{3.45}$$

Actually, we have established this last inequality, with $U_n(\pm 1+iy)$ replaced by

$$\int_0^1 \log\{1+(y/(1-t))^2\}\,\mu_n(t)\,dt$$

$$= \limsup_{x \to 1-} \int_0^1 \log\{1+(y/(x-t))^2\}\,\mu_n(t)\,dt$$

for we first estimated this second integral in the proof of (a). Since for $|x|>1$,

$$U_n(x+iy) \leqslant \int_0^1 \log\{1+(y/(|x|-t))^2\}\,\mu_n(t)\,dt$$

$$\leqslant \int_0^1 \log\{1+(y/(1-t))^2\}\,\mu_n(t)\,dt,$$

we obtain (3.45) with x replacing 1. Finally, the bound for A_n^*, used in (3.40), appears in (3.28). ∎

We need one more estimate involving $U_n(x)$ for x larger than 1:

LEMMA 3.5. *Let $W(x)$ be as in Lemma 2.1, with the additional restrictions that $Q''(x)$ is continuous in $\mathbb{R}$, and that (2.23) is satisfied for some $0<\eta<\frac{1}{3}$. Let $m=m(n)$, n large enough, be such that*

$$\lim_{n\to\infty} m^{(1-3\eta)/(1-\eta)}/(n\log m)=\infty. \tag{3.46}$$

Then there exist C_1 and C_2 such that for $s\geqslant a_m/a_n$, and $n\geqslant C_1$,

$$Q'(a_n s)\exp(nU_n(s))\leqslant \exp(-m^{(1-3\eta)/(1-\eta)}). \tag{3.47}$$

Proof. Now from Lemma 3.1(c),

$$\begin{aligned}
U_n(s) &= U_n(s)-U_n(0)\\
&=\int_{-1}^{1}\log|s-t|\,\mu_n(t)\,dt\\
&\quad -\int_{-1}^{1}\log|t|\,\mu_n(t)\,dt-Q(a_n s)/n+Q(0)/n\\
&\leqslant \log(s+1)+C_3\int_{-1/2}^{1/2}\log\left(\frac{1}{t}\right)dt\\
&\quad +\log 4\int_{1/2}^{1}\mu_n(t)\,dt-Q(a_n s)/n+C_4\\
&\leqslant \log(s+1)-Q(a_n s)/n+C_5,
\end{aligned} \tag{3.48}$$

where we have used Lemma 3.2(a). Next, since a_u is a positive strictly increasing and continuous function of u, our bound $s\geqslant a_m/a_n$ ensures that we can write $a_n s=a_l$, where $l\geqslant m$. Then, from Lemma 2.2(g),

$$\log(s+1)=\log(a_l/a_n+1)\leqslant \log l,$$

for $n\geqslant C_1$, where C_1 is independent of s and n. Further, by Lemma 2.3(a),

$$\log Q'(a_n s)=\log Q'(a_l)\leqslant C\log l,$$

where C is independent of n and s. Using (3.48), we have for $n\geqslant C_1$ and $a_n s=a_l\geqslant a_m$ that

$$Q'(a_n s)\exp(nU_n(s))\leqslant \exp(C_6 n\log l+C_7 n-Q(a_l)). \tag{3.49}$$

Here, as $Q''(x)\geqslant 0$ for x large enough, we have

$$\begin{aligned}
Q(a_l) &\geqslant Q(a_{l/2})+Q'(a_{l/2})(a_l-a_{l/2})\\
&\geqslant Q'(a_{l/2})\,a_l(1-a_{l/2}/a_l)\\
&\geqslant Q'(a_{l/2})\,a_{l/2}(C_8/\chi(a_l)) \qquad \text{(by Lemma 2.2(e))}\\
&\geqslant l^{1-2\eta/(1-\eta)},
\end{aligned}$$

by Lemma 2.2(a) (with $j=1$) and by Lemma 2.3(a), provided n is large enough. Then (3.46) and (3.49), and the fact that $l \geqslant m$, easily yield (3.47). ∎

4. Proof of Theorems 1.3 and 1.5

Our main lemma for estimating $(PW)'$ follows:

LEMMA 4.1. *Let $W(x) := e^{-Q(x)}$ be as in Lemma 2.1. Assume in addition that $Q(0)=0$ and for some $1<p<2$, (3.1) is satisfied, and let $U_n(z)$ be defined by (3.11). Then if $s \in (0, \infty)$, $\varepsilon \in (0, 1)$, $n \geqslant 1$, and $P \in \mathscr{P}_n$,*

$$|(PW)'(a_n s)| \leqslant \|PW\|_{\mathbb{R}} (\varepsilon a_n)^{-1} \{ \max_{|t-s|=\varepsilon} \exp(nU_n(t))\} e^{\tau}, \tag{4.1}$$

where for some C,

$$\tau := \begin{cases} 4[a_n s Q'(a_n s)\{\varepsilon/(s-\varepsilon)\}^2 + (a_n \varepsilon)^2 Q''(a_n(s+\varepsilon))], \\ \qquad \text{if} \quad a_n(s-\varepsilon) \geqslant C, \\ [Q(a_n(s+\varepsilon)) + \varepsilon a_n Q'(a_n s)], \\ \qquad \text{if} \quad a_n(s-\varepsilon) < C. \end{cases} \tag{4.2}$$

If, in addition, Q' is continuous at 0, then (4.1) holds also for $s=0$.

Proof. For fixed $s \in (0, \infty)$, define a new weight $\hat{W}(t) := e^{-\hat{Q}(t)}$, where $\hat{Q}(t)$ is the linear function

$$\hat{Q}(t) := Q(a_n s) + Q'(a_n s)(t - a_n s), \qquad t \in \mathbb{C}. \tag{4.3}$$

Note that $\hat{W}$ is an entire function, and

$$\hat{W}^{(j)}(a_n s) = W^{(j)}(a_n s), \qquad j = 0, 1. \tag{4.4}$$

Then if $P \in \mathscr{P}_n$,

$$(PW)'(a_n s) = (P\hat{W})'(a_n s) = (2\pi i)^{-1} \int_\Gamma \frac{P\hat{W}(z)}{(z-a_n s)^2}\, dz,$$

where Γ is the circle $\{z\colon |z - a_n s| = a_n \varepsilon\}$, and we have used Cauchy's integral formula for derivatives. Then we obtain

$$\begin{aligned} |(PW)'(a_n s)| &\leqslant \max_{z\in\Gamma} |P\hat{W}(z)|\, (\varepsilon a_n)^{-1} \\ &\leqslant \max_{|t-s|=\varepsilon} |P(a_n t)\, W(a_n |t|)| \max_{|t-s|=\varepsilon} |\hat{W}(a_n t)/W(a_n |t|)|\, (\varepsilon a_n)^{-1} \\ &\leqslant \|PW\|_{\mathbb{R}} (\varepsilon a_n)^{-1} \{ \max_{|t-s|=\varepsilon} \exp(nU_n(t))\}\, \rho, \end{aligned} \tag{4.5}$$

by Lemma 3.1(d) and with

$$\rho := \max_{|t-s|=\varepsilon} |\hat{W}(a_n t)/W(a_n|t|)|.$$

It remains to estimate ρ. Suppose first that $a_n(s-\varepsilon) \geqslant C$, where C is so large that Q'' is positive and increasing in $[C, \infty)$. Let $|t-s|=\varepsilon$ and write $t=|t|\,e^{i\theta}$, some $\theta \in [-\pi, \pi)$. Then, for some v between $|t|$ and s,

$$\begin{aligned}
&|\hat{W}(a_n t)/W(a_n|t|)| \\
&\quad = \exp[-Q(a_n s) - Q'(a_n s)\, a_n(\operatorname{Re} t - s) + Q(a_n|t|)] \\
&\quad = \exp[-Q(a_n s) - Q'(a_n s)\, a_n(\operatorname{Re} t - s) + Q(a_n s) \\
&\qquad + Q'(a_n s)\, a_n(|t|-s) + a_n^2 Q''(a_n v)(|t|-s)^2/2] \\
&\quad = \exp[a_n Q'(a_n s)\, |t|\, (1-\cos\theta) + a_n^2 Q''(a_n v)\, (|t|-s)^2/2] \\
&\quad \leqslant \exp[a_n Q'(a_n s)(s+\varepsilon)\, \theta^2/2 + a_n^2 Q''(a_n v)\, \varepsilon^2/2],
\end{aligned} \tag{4.6}$$

by the inequality

$$1-\cos\theta \leqslant \theta^2/2, \qquad \theta \in [-\pi, \pi].$$

Next, $\operatorname{Re} t \geqslant s-\varepsilon \geqslant C/a_n$, so $|\theta| \in [0, \pi/2]$, and we have

$$\frac{2}{\pi}\,|\theta| \leqslant |\sin\theta| = \frac{|\operatorname{Im} t|}{|t|} \leqslant \frac{\varepsilon}{s-\varepsilon},$$

so

$$a_n Q'(a_n s)(s+\varepsilon)\, \theta^2/2 \leqslant 4 a_n s Q'(a_n s) \{\varepsilon/(s-\varepsilon)\}^2,$$

while the monotonicity of Q'' yields

$$a_n^2 Q''(a_n v)\, \varepsilon^2/2 \leqslant a_n^2 Q''(a_n(s+\varepsilon))\, \varepsilon^2.$$

Hence, from (4.6),

$$\rho \leqslant \exp[4 a_n s Q'(a_n s)\{\varepsilon/(s-\varepsilon)\}^2 + a_n^2 Q''(a_n(s+\varepsilon))\, \varepsilon^2],$$

and then (4.5) yields (4.1) and (4.2).

If $a_n(s-\varepsilon) < C$, then for $|t-s| = \varepsilon$,

$$\begin{aligned}
&|\hat{W}(a_n t)/W(a_n|t|)| \\
&\quad = \exp[-Q(a_n s) - Q'(a_n s)\, a_n(\operatorname{Re} t - s) + Q(a_n|t|)] \\
&\quad \leqslant \exp[Q(a_n(s+\varepsilon)) + Q'(a_n s)\, \varepsilon a_n],
\end{aligned}$$

since $Q(x) > Q(0) = 0$, for $x > 0$. ∎

Proof of Theorem 1.3 *in a Special Case.* Suppose first that $W(x)$ is as in Lemma 2.1, with the additional restrictions that $Q''(x)$ is continuous in $\mathbb{R}$ and that (1.12) holds. We may also assume that $Q(0)=0$—if not, replace $W(x)$ by $W(x)/W(0)=e^{Q(x)-Q(0)}$. Such a replacement clearly does not affect (1.13). Note that then the requirements of Lemmas 2.1, 2.2, 3.1, 3.3, 3.4, 4.1 are satisfied, as are those of Lemmas 2.3 and 3.5, with $\eta=\frac{1}{4}$. By (3.19) in Lemma 3.1(d), for $P\in\mathscr{P}_n$ and $n\geqslant 1$,

$$\begin{aligned}\|P'W\|_{\mathbb{R}} &= \max_{s\in[-1,1]} |(P'W)(a_ns)| \\ &= \max_{s\in[-1,1]} |(PW)'(a_ns)+Q'(a_ns)(PW)(a_ns)| \\ &\leqslant \max_{s\in[0,1]} \{e^{\tau} \max_{|t-s|=\varepsilon} \exp(nU_n(t))\} \|PW\|_{\mathbb{R}} (\varepsilon a_n)^{-1} \\ &\quad + CQ'(a_n)\|PW\|_{\mathbb{R}}, \end{aligned} \tag{4.7}$$

by (2.12), by the evenness of W, and by Lemma 4.1 with the notation there. We set

$$\varepsilon := \varepsilon(n) := 1/\{a_nQ'(a_n)\}.$$

By Lemma 3.3(c), we have, uniformly for $s\in[0,1]$,

$$\begin{aligned}\max_{|t-s|=\varepsilon} \exp(nU_n(t)) &\leqslant \max_{|t-s|=\varepsilon} \exp\{Ca_nQ'(a_n)\,|\mathrm{Im}\,t|\} \\ &\leqslant \exp\{Ca_nQ'(a_n)\varepsilon\} \leqslant C_3. \end{aligned} \tag{4.8}$$

It remains to estimate τ, given by (4.2). Suppose first $a_n(s-\varepsilon)<C$. Then

$$0<a_n(s+\varepsilon)<C+2\varepsilon a_n<C_4,$$

so the continuity of Q and Q' and (4.2) yield uniformly for such s and for $n\geqslant 1$ that

$$\tau\leqslant C_5. \tag{4.9}$$

Suppose next that $a_n(s-\varepsilon)\geqslant C$, where (as in the proof of Lemma 4.1) C is so large that $Q''(x)$ is positive and increasing for $x\geqslant C$. Then from (4.2),

$$\begin{aligned}\tau &\leqslant 4[a_nQ'(a_n)\,\varepsilon^2(C/a_n)^{-2}+(a_n\varepsilon)^2\,Q''(a_n(1+\varepsilon))] \\ &\leqslant 4[a_nQ'(a_n)^{-1}\,C^{-2}+Q'(a_n)^{-2}\,Q''(a_n\{1+o(n)^{-1}\})], \end{aligned}$$

by choice of ε, and by Lemma 2.2(a), with $j=1$. Combining Lemma 2.2(a) with $j=1$, Lemma 2.2(g), and (2.31) of Lemma 2.3(c) (recall that $\eta=\frac{1}{4}$ in our case), we obtain

$$\tau \leqslant 4[o(1)+o((a_n/n)^2)\,O((n/a_n)^2]=o(1),$$

so (4.9) remains valid. Then (4.7) to (4.9) yield (1.13). ∎

Proof of Theorem 1.3 *in the General Case.* Suppose now that W satisfies the conditions of Theorem 1.3. We shall redefine $W(x)$ for small x, obtaining a new weight $W^*(x):=e^{-Q^*(x)}$, where Q^* is twice continuously differentiable in $\mathbb{R}$, and W^* satisfies the conditions of Lemma 2.1 and (1.12). Let ε be a small positive number, let

$$L(x):=\{x^2+\varepsilon(x^2-\rho^2)^4\}^{1/2}, \qquad x\in[-\rho,\rho],$$

and let

$$Q^*(x):=\begin{cases} Q(L(x)), & x\in[-\rho,\rho],\\ Q(x), & |x|>\rho.\end{cases}$$

Then $Q^*(x)$ is even and twice continuously differentiable in $(-\rho,\rho)$ since $L(x)$ is bounded below there by a positive number. As

$$L(\rho)=\rho; \qquad L'(\rho)=1; \qquad L''(\rho)=0,$$

we see that $Q^{*\prime\prime}(x)$ is continuous at ρ and so continuous in $\mathbb{R}$. Next, we see that for $x\in[-\rho,\rho]$,

$$\frac{xL'(x)}{L(x)}=\left(\frac{x}{L(x)}\right)^2\{1+4\varepsilon(x^2-\rho^2)^3\}, \tag{4.10}$$

and

$$\frac{xL''(x)}{L'(x)}=1-\left(\frac{x}{L(x)}\right)^2+\varepsilon x^2(x^2-\rho^2)^2\,g(x), \tag{4.11}$$

where

$$g(x):=\frac{24}{1+4\varepsilon(x^2-\rho^2)^2}+\frac{4(\rho^2-x^2)}{L(x)^2}.$$

As $g(x)$ is positive and continuous in $[-\rho,\rho]$, and as

$$|x|/L(x)\leqslant 1, \qquad x\in[-\rho,\rho],$$

we see that if ε is small enough,

$$L^{(j)}(x)>0, \qquad x\in(0,\rho),\ j=1,2.$$

Then (2.1) holds for Q^*. Further, a straightforward calculation shows that for $x \in [-\rho, \rho]$,

$$\chi^*(x) := (xQ^{*\prime}(x))'/Q^{*\prime}(x)$$
$$= 1 + \frac{xL'(x)}{L(x)}\chi(L(x)) + \frac{xL''(x)}{L'(x)} - \frac{xL'(x)}{L(x)},$$

while for $x \in [\rho, \infty)$, $\chi^*(x) = \chi(x)$ is positive and increasing. If we can show that $\chi^*(x)$ is positive and continuous in $[0, \rho]$, then it will follow that $\chi^*(x)$ is quasi-increasing in $[0, \infty)$, and the remaining requirements of Lemma 2.1 (including (2.2)) will follow. Using (4.10), (4.11), the definition of g, and some manipulations, we obtain for $x \in [0, \rho]$ that

$$\chi^*(x) = 2\left\{1 - \left(\frac{x}{L(x)}\right)^2\right\} + \frac{xL'(x)}{L(x)}\chi(L(x))$$
$$+ \varepsilon x^2(x^2 - \rho^2)^2\left[g(x) + \frac{4(\rho^2 - x^2)}{L(x)^2}\right].$$

The first of the three terms in this last right-hand side is positive for $x \in [0, \rho)$. The second term is positive for $x \in (0, \rho]$ provided ε is small enough. Finally, the third term is positive in $(0, \rho)$, provided ε is small enough. Hence we can ensure that

$$\min\{\chi^*(x) \colon x \in [0, \rho]\} > 0.$$

As W^* fulfills all the requirements for the special case of Theorem 1.3 proved above, (1.13) holds for W^*. As

$$W(x) \sim W^*(x),\ x \in \mathbb{R}; \qquad Q(x) = Q^*(x),\ |x| > \rho,$$

we have

$$\|P'W\|_{\mathbb{R}} \leqslant CQ'(a_n^*)\,\|PW\|_{\mathbb{R}}, \qquad P \in \mathscr{P}_n,\ n \geqslant C_1, \tag{4.12}$$

where a_n^* is the root of (1.7) for Q^*. It remains to show that

$$Q'(a_n^*) \sim Q'(a_n), \qquad n \text{ large enough.} \tag{4.13}$$

(For $n \leqslant C_1$, (1.13) follows easily from a compactness argument, and the positivity of $Q'(a_n)$, $1 \leqslant n < C_1$.) Now from (1.7) for a_n^* and a substitution,

$$n = \frac{2}{\pi}\left\{\frac{1}{a_n^*}\int_0^\rho \frac{uQ^{*\prime}(u)}{(1-(u/a_n^*)^2)^{1/2}}\,du + \int_{\rho/a_n^*}^1 \frac{a_n^* tQ'(a_n^* t)}{(1-t^2)^{1/2}}\,dt\right\}$$
$$= O(1/a_n^*) + \frac{2}{\pi}\int_0^1 \frac{a_n^* tQ'(a_n^* t)}{(1-t^2)^{1/2}}\,dt.$$

We deduce that for n large enough,

$$n-1 \leqslant \frac{2}{\pi} \int_0^1 \frac{a_n^* t Q'(a_n^* t)}{(1-t^2)^{1/2}}\, dt \leqslant n+1.$$

The monotonicity and positivity of $sQ'(s)$ in $(0, \infty)$ then yield

$$a_{n-1} \leqslant a_n^* \leqslant a_{n+1}.$$

Since W itself satisfies the conditions of Lemma 2.1, and satisfies (2.23) with $\eta = \frac{1}{4}$, we may use Lemma 2.3(b) with $m := n+1$ to deduce that

$$\lim_{n \to \infty} Q'(a_{n+1})/Q'(a_{n-1}) = 1,$$

and hence

$$\lim_{n \to \infty} Q'(a_n^*)/Q'(a_n) = 1. \quad \blacksquare$$

We shall prove Theorem 1.5 in several stages. The first lemma treats $|x| \leqslant (1-\eta)\, a_n$, $\eta \in (0, 1)$ fixed. As remarked after Theorem 1.3 (remark (vii)), a result more general than Lemma 4.2 was proved using simpler Christoffel function methods in [13, Corollary 3.5], but we include the proof for the sake of completeness.

LEMMA 4.2. *Let $W(x)$ be as in Theorem 1.5. Let $0 < \eta < 1$. Then for $n \geqslant C_1$, $P \in \mathscr{P}_n$, and $|x| \leqslant (1-\eta)\, a_n$,*

$$|(PW)'(x)| \leqslant C_2(n/a_n)\, \|PW\|_{\mathbb{R}}. \tag{4.14}$$

Proof. Suppose first that Q'' is continuous in $\mathbb{R}$. Then for $|x| \leqslant a_n(1-\eta)$, we can write $x = a_n s$, where $|s| \leqslant 1-\eta$. Since W is even, it suffices to consider $s \in [0, 1-\eta]$. Let

$$\varepsilon := \varepsilon(n) := n^{-1}, \qquad n \geqslant 1.$$

Lemma 4.1 yields

$$\begin{aligned} |(PW)'(x)| &= |(PW)'(a_n s)| \\ &\leqslant \|PW\|_{\mathbb{R}}(n/a_n)\, e^{\tau} \max_{|t-s| = 1/n} \exp(nU_n(t)), \end{aligned}$$

where τ depends on n and s, and is given by (4.2). Lemma 3.3(b) shows that

$$\max_{|t-s| = 1/n} \exp(nU_n(t)) \leqslant \max_{|t-s| = 1/n} \exp(nC|\operatorname{Im} t|) \leqslant C_3.$$

It remains to estimate τ. If $a_n(s-\varepsilon)<C$, we can show that (4.9) holds exactly as at (4.9). If $a_n(s-\varepsilon)\geqslant C$, we see from (2.12) with $j=2$, from (4.2), and from the monotonicity of $uQ'(u)$, that for n large enough and $s\in[0, 1-\eta]$,

$$\tau\leqslant C_4[a_n(1-\eta)Q'(a_n(1-\eta))(a_n/(nC))^2+(a_n/n)^2 Q''(a_n(1-\eta/2))]$$
$$=o(1),$$

by Lemma 2.2(b) and (g). This completes the proof for the case where Q'' is continuous in $\mathbb{R}$. In the general case, we replace Q by Q^* as in the previous proof, and use the boundedness of $Q^{*\prime}$ and Q' in each finite interval, as well as the fact that

$$W\sim W^*; \qquad a_n\sim a_n^*. \quad \blacksquare$$

LEMMA 4.3. *Let $W(x)$ be as in Theorem* 1.5. *Let $r>0$. Then for $n\geqslant C_1$, $P\in\mathscr{P}_n$, and*

$$\eta\leqslant|x/a_n|\leqslant 1-r(nA_n^*)^{-2/3}, \tag{4.15}$$

we have

$$|(PW)'(x)|\leqslant C(1-|x/a_n|)^{-1}$$
$$\times\int_{|x/a_n|}^{1}\psi_n(t)(1-t)^{1/2}\,dt\,\|PW\|_{\mathbb{R}}. \tag{4.16}$$

Proof. We assume first that Q'' is continuous in $\mathbb{R}$. Recall from Lemma 2.3 with $\eta=\frac{1}{24}$ that, as $n\to\infty$,

$$Q'(a_n)=O((n/a_n)^{24/23}), \tag{4.17}$$

$$\chi(a_n)=O((n/a_n)^{2/23}), \tag{4.18}$$

and

$$a_nQ''(a_n)=O((n/a_n)^{26/23}). \tag{4.19}$$

Then for $n\geqslant C_1$,

$$1-r(nA_n^*)^{-2/3}\geqslant 1-rn^{-2/3}\geqslant 1-r\chi(a_n)^{-15/2}.$$

Hence Lemma 3.2(c) and (g) yield

$$\mu_n(t)\sim A_n^*(1-t)^{1/2}, \qquad 1>t\geqslant 1-r(nA_n^*)^{-2/3}, \tag{4.20}$$

and so for $n \geqslant C_1$

$$\int_t^1 \mu_n(y)\,dy \sim A_n^*(1-t)^{3/2}, \qquad 1 > t \geqslant 1 - r(nA_n^*)^{-2/3}. \tag{4.21}$$

Now set for some fixed $\lambda > 0$,

$$\varepsilon := \varepsilon(n, s) := \left[\lambda n\, \delta(s)^{-1} \int_s^1 \mu_n(t)\,dt \right]^{-1}, \tag{4.22}$$

where

$$s := x/a_n \in [\eta, 1 - r(nA_n^*)^{-2/3}], \tag{4.23}$$

and

$$\delta(s) := (1-s)/2. \tag{4.24}$$

(Note that, as usual, we may restrict ourselves to $x > 0$). We first derive several upper bounds for ε. First, from (4.21) and (4.23),

$$\int_s^1 \mu_n(t)\,dt \geqslant \int_{1-r(nA_n^*)^{-2/3}}^1 \mu_n(t)\,dt \sim A_n^*(nA_n^*)^{-1} = n^{-1}.$$

Then

$$\varepsilon \leqslant [\lambda n\, \delta(s)^{-1}\, C_2 n^{-1}]^{-1} \leqslant \delta(s)/2, \tag{4.25}$$

provided $\lambda \geqslant 2/C_2$. Next, from Lemma 3.2(c), (d), and (e),

$$\begin{aligned}\int_s^1 \mu_n(t)\,dt &\geqslant C_3(a_n/n)\, \psi_n(s)(1-s)^{3/2} \\ &\geqslant C_4(a_n/n)\, \psi_n(\eta/2)\, \delta(s)^{3/2} \\ &\geqslant C_5\, \delta(s)^{3/2},\end{aligned}$$

so

$$\begin{aligned}\varepsilon \leqslant C_6 n^{-1}\, \delta(s)^{-1/2} &\leqslant C_7 n^{-2/3} A_n^{*\,1/3} \\ &\leqslant C_8 n^{-44/69} = o(n^{-1/2}),\end{aligned} \tag{4.26}$$

by Lemma 3.2(f) and (4.18). Finally, using Lemma 3.2(b), we obtain, much as above,

$$\int_s^1 \mu_n(t)\,dt \geqslant C_9(a_n/n)\, \delta(s)^2\, \{a_n s Q''(a_n s) + Q'(a_n s)\},$$

and hence

$$\varepsilon \leqslant C_{10}\, \delta(s)^{-1}\, \{a_n^2 s Q''(a_n s) + a_n Q'(a_n s)\}^{-1}. \tag{4.27}$$

Now let $|t-s|=\varepsilon$, and write $\operatorname{Re} t = s+\Delta$, where $\Delta \in [-\varepsilon, \varepsilon]$. We see that

$$\delta(\operatorname{Re} t) = \delta(s) - \Delta/2 \in (\delta(s)/2, 3\delta(s)/2),$$

by (4.25). Also,

$$\operatorname{Re} t + \delta(\operatorname{Re} t) \geqslant s - \varepsilon + \delta(s)/2 \geqslant s.$$

Then Lemma 3.4(a) yields

$$\begin{aligned} nU_n(t) &\leqslant C_2 \left\{ n|\operatorname{Im} t|^2 + \left[\frac{n|\operatorname{Im} t|}{\delta(\operatorname{Re} t)} \int_{\operatorname{Re} t + \delta(\operatorname{Re} t)}^1 \mu_n(t)\, dt \right] \right. \\ &\quad \left. \times \left[1 + \left\{ \frac{|\operatorname{Im} t|}{\delta(\operatorname{Re} t)} \right\}^{1/2} \right] \right\} \\ &\leqslant C_2 \left\{ n\varepsilon^2 + \left[\frac{2n\varepsilon}{\delta(s)} \int_s^1 \mu_n(t)\, dt \right] \left[1 + \left\{ \frac{2\varepsilon}{\delta(s)} \right\}^{1/2} \right] \right\} \\ &\leqslant C_2 \{ o(1) + O(1) \}, \end{aligned}$$

by (4.22), (4.25), and (4.26). Next, as $s+\delta(s) = (1+s)/2 < 1$, (4.2) shows that for $n \geqslant C_1$,

$$\begin{aligned} \tau &\leqslant 4[a_n s Q'(a_n s)(2\varepsilon/\eta)^2 + (a_n \varepsilon)^2 Q''(a_n)] \\ &\leqslant C_{11}[\varepsilon/\delta(s) + n^{-88/69} a_n^2 Q''(a_n)] \\ &\qquad \text{(by (4.26) and (4.27))} \\ &\leqslant C_{13}[\tfrac{1}{2} + n^{-88/69+26/23}] \leqslant C_{14}, \end{aligned}$$

by (4.19) and (4.25). These last estimates and Lemma 4.1 yield

$$|(PW)'(a_n s)| \leqslant \|PW\|_{\mathbb{R}}\, C_{15}\, \delta(s)^{-1} (n/a_n) \int_s^1 \mu_n(t)\, dt,$$

and then Lemma 3.2(c) yields the lemma. Finally, if Q'' is not continuous at 0, we replace Q by Q^*, as before. For n large enough, A_n^* for Q and Q^* are identical, while if ξ in the definition of $\psi_n(x)$ is large enough, $\psi_n(x)$ for Q and Q^* are identical. It is not difficult to use the estimates of Lemma 3.2(d) and (e) to show that increasing ξ by a fixed amount has little effect on ψ_n, since $\xi > 0$ in Lemma 3.2 was arbitrary. ▌

Finally, we deal with x near a_n:

LEMMA 4.4. *Let $W(x)$ be as in Theorem* 1.5, *and let $r>0$, and for $n \geqslant 1$, let*

$$m := m(n) := n^{23/20}. \tag{4.28}$$

Then for $n \geqslant C_1$, $P \in \mathscr{P}_n$, and

$$1 - r(nA_n^*)^{2/3} \leqslant |x/a_n| \leqslant a_m. \tag{4.29}$$

we have

$$|(PW)'(x)| \leqslant C(nA_n^*)^{2/3} a_n^{-1} \|PW\|_{\mathbb{R}}. \tag{4.30}$$

Proof. As above, we can assume that Q'' is continuous in $\mathbb{R}$. Let

$$s := x/a_n \in [1 - r(nA_n^*)^{-2/3}, a_m/a_n],$$

and

$$\varepsilon := \varepsilon(n) := (nA_n^*)^{-2/3}.$$

Let $|t-s| = \varepsilon$. If $\operatorname{Re} t \geqslant 1$, Lemma 3.4(b) shows that

$$nU_n(t) \leqslant CnA_n^* |\operatorname{Im} t|^{3/2} \leqslant CnA_n^* \varepsilon^{3/2} = C.$$

If $\operatorname{Re} t < 1$, then as $\operatorname{Re} t \geqslant s - \varepsilon \geqslant 1 - (r+1)(nA_n^*)^{-2/3}$, Lemma 3.4(a) and (4.21) yield

$$\begin{aligned} nU_n(t) &\leqslant C_2 \left\{ n|\operatorname{Im} t|^2 + \left[\frac{n|\operatorname{Im} t|}{\delta(\operatorname{Re} t)} \int_{\operatorname{Re} t + \delta(\operatorname{Re} t)}^{1} \mu_n(t)\, dt \right] \right. \\ &\qquad \left. \times \left[1 + \left\{ \frac{|\operatorname{Im} t|}{\delta(\operatorname{Re} t)} \right\}^{1/2} \right] \right\} \\ &\leqslant C_3 \left\{ n\varepsilon^2 + [n\varepsilon A_n^* \,\delta(\operatorname{Re} t)^{1/2}] \left[1 + \left\{ \frac{\varepsilon}{\delta(\operatorname{Re} t)} \right\}^{1/2} \right] \right\} \\ &\leqslant C_4 \{ n^{-1/3} + n\varepsilon A_n^* \,\delta(\operatorname{Re} t)^{1/2} + n\varepsilon^{3/2} A_n^* \}. \end{aligned}$$

Since $\delta(\operatorname{Re} t) \leqslant ((r+1)/2)(nA_n^*)^{-2/3}$, we obtain

$$nU(t) \leqslant C_5, \qquad |t-s| = \varepsilon.$$

Next, we estimate τ given by (4.2). Recall from (4.18) that

$$\chi(a_{2m}) = O((2m/a_{2m})^{2/23}) = o(n^{1/10}),$$

so for $n \geqslant C_1$,

$$a_n(s+\varepsilon) \leqslant a_m + o(a_n n^{-2/3}) \leqslant a_m\{1 + o(\chi(a_{2m})^{-1})\} \leqslant a_{2m},$$

by Lemma 2.2(e). Then we have for $n \geqslant C_1$ that

$$\begin{aligned}\tau &\leqslant 4\{a_m Q'(a_m)(2\varepsilon)^2 + (a_n\varepsilon)^2 Q''(a_{2m})\} \\ &\leqslant \{o(m^{24/23})o(n^{-4/3}) + o(n^{-4/3})o(m^{26/23})\} \\ &= O(n^{-1/30}),\end{aligned}$$

by (4.19), (4.19), and the choice (4.28) of m. The above estimates and Lemma 4.1 immediately yield (4.30). ▌

Proof of Theorem 1.5. Assume first that Q'' is continuous in $\mathbb{R}$. Note that if $0<\delta<1$, and $|x/a_n| \leqslant 1-\delta$, then Lemma 3.2(c) and (d) show that

$$\begin{aligned}&(1-|x/a_n|)^{-1} \int_{|x/a_n|}^{1} \psi_n(t)(1-t)^{1/2}\,dt \\ &\sim 1 \times \left[\int_{|x/a_n|}^{1-\delta/2} (n/a_n)\,dt + \int_{1-\delta/2}^{1} (n/a_n)\,\mu_n(t)\,dt\right] \sim n/a_n.\end{aligned}$$

Then Lemmas 4.2 and 4.3 yield the conclusion of Theorem 1.5 for $|x/a_n| \leqslant 1 - r(nA_n^*)^{-2/3}$. For the range (4.29), with m as in (4.28), Lemma 4.4 yields the desired conclusion. It remains to deal with $x > a_m$, and we use Lemma 3.5, with $\eta = \frac{1}{24}$. Note that

$$m^{(1-3\eta)/(1-\eta)}/n = m^{21/23}/n = n^{1/20} \to \infty, \qquad n \to \infty, \qquad \text{as} \quad n \to \infty,$$

that is, the requirement of Lemma 3.5 is fulfilled. Write $x = a_n s$, where $s > a_m/a_n > 1$. We have for $P \in \mathscr{P}_n$, from Lemma 3.1(d),

$$\begin{aligned}|(PW)'(x)| &\leqslant |P'W|(x) + Q'(x)\,|PW|(x) \\ &\leqslant \|P'W\|_{\mathbb{R}} \exp(nU_n(s)) + Q'(x)\,\|PW\|_{\mathbb{R}} \exp(nU_n(s)) \\ &\leqslant \exp(nU_n(s))\,\|PW\|_{\mathbb{R}}\,\{CQ'(a_n) + Q'(x)\} \qquad \text{(by Theorem 1.3)} \\ &\leqslant C_2 Q'(a_n s)\exp(nU_n(s))\,\|PW\|_{\mathbb{R}} \\ &\leqslant C_3 \exp(-m^{21/23})\,\|PW\|_{\mathbb{R}},\end{aligned}$$

by Lemma 3.5, and choice of m. This proves somewhat more than the conclusion of Theorem 1.5. Finally, in the case that Q'' is not continuous at 0, we replace Q by Q^*, as usual. ▌

Note added in proof. After completion of this paper, the limit (1.19) has been established, under mild additional conditions on Q. Hence $Q'(a_n)$ in Theorem 1.3 is sharp. See Theorem 2.6 in "Strong Asymptotics for Extremal Errors and Polynomials Associated with Erdős Weights," Pitman Research Notes, Volume 202, Longmans, London, 1989.

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